COVARIANT SUPERGRAPHS II

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Prerequisite and aim

We have seen that for special background chiral and vector multiplets,

$$[\Phi, \bar{\Phi}] \;=\; \mathcal{D}^\alpha \mathcal{W}_\alpha = 0 \;, \qquad \bar{\mathcal{D}}_{\dot\alpha} \Phi \;=\; \mathcal{D}_\alpha \Phi \;=\; 0 \;,$$

all the propagators are expressed via a single Green's functions $G(z, z')$ (chosen in different representations of the gauge group):

$$(\Box_{\rm v} - |\mathcal{M}|^2)\, G(z, z') = -\mathbf{1}\, \delta^8(z - z') \;.$$

Here the delta-function and the vector d'Alembertian are

$$\delta^8(z - z') = \delta^4(x - x')\,(\theta - \theta')^2 (\bar\theta - \bar\theta')^2 \;,$$
$$\Box_{\rm v} = \mathcal{D}^a \mathcal{D}_a - \mathcal{W}^\alpha \mathcal{D}_\alpha + \bar{\mathcal{W}}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} \;.$$

Finally, the mass operator $\mathcal{M}$ is defined by $\mathcal{M}_{\rm R}\,\Sigma = -{\rm i}\,\Phi\,\Sigma$, for a multiplet Σ transforming in an arbitrary representation R of the gauge group.

In this lecture, we will study more general (but related) situation: (i) *arbitrary* background vector multiplet; (ii) $|\mathcal{M}|^2 \;\to m^2 \mathbf{1}$.
Our aim will consist in developing a covariant expansion of the corresponding propagator in powers of the Yang-Mills superfield strengths $\mathcal{W}_\alpha$ and $\bar{\mathcal{W}}_{\dot\alpha}$, and their covariant derivatives.

The presentation follows mainly SMK, McArthur (2003)

Covariant derivative expansion in Yang-Mills theory (Non-supersymmetric case)

Consider a Green's function,

$$G^i{}_{i'}(x,x') = \mathrm{i}\,\langle \varphi^i(x)\,\bar{\varphi}_{i'}(x')\rangle \ ,$$

associated with a quantum field $\varphi = (\varphi^i(x))$, which transforms in some representation of the gauge group $\mathbf{G}$, and its conjugate $\varphi^\dagger = (\bar{\varphi}_i(x))$. The Green's function satisfies the equation

$$\Delta_x\, G(x,x') = -\delta^d(x-x')\,\mathbf{1} \ ,$$
$$\Delta = \nabla^m \nabla_m - \mathcal{U} \ , \qquad \mathbf{1} = (\delta^i{}_{i'}) \ ,$$

with ∇_m the gauge-covariant derivatives,

$$\nabla_m = \partial_m + \mathrm{i}\,A_m \ , \qquad [\nabla_m, \nabla_n] = \mathrm{i}\,F_{mn} \ , \qquad A_m = A^I_m(x)\,T_I \ ,$$

and $\mathcal{U}(x)$ a local matrix function of the background field containing a mass term $m^2\,\mathbf{1}$.

`Gauge transformation:`

$$\nabla_m \ \to \ \mathrm{e}^{\mathrm{i}\tau(x)}\,\nabla_m\,\mathrm{e}^{-\mathrm{i}\tau(x)} \ , \quad \varphi \ \to \ \mathrm{e}^{\mathrm{i}\tau(x)}\,\varphi \ , \quad \mathcal{U} \ \to \ \mathrm{e}^{\mathrm{i}\tau(x)}\,\mathcal{U}\,\mathrm{e}^{-\mathrm{i}\tau(x)} \ ,$$

and therefore

$$G(x,x') \ \to \ \mathrm{e}^{\mathrm{i}\tau(x)}\,G(x,x')\,\mathrm{e}^{-\mathrm{i}\tau(x')} \ ,$$

with $\tau = \tau^I(x)\,T_I = \tau^\dagger$.

Parallel transporter

Let $\gamma(t)$ be a curve connecting two points, x and x'.

$$\gamma: \ [0,1] \ \rightarrow \mathbb{R}^{d-1,1} \ , \qquad \gamma(0) = x' \ , \quad \gamma(1) = x \ .$$

Introduce the *operator of parallel transport* (also known as Schwinger's phase factor or Wilson's line), $I_\gamma(t)$, along the curve,

$$I_\gamma(t): \ [0,1] \ \rightarrow \mathbf{G} \ , \quad I_\gamma(0) = \mathbf{1} \ ,$$
$$\Big(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\, \dot{x}^m(t)\, A_m(t)\Big)\, I_\gamma(t) = 0 \ ,$$

with $\mathbf{G}$ the gauge group. We have

$$I_\gamma(x,x') = I_\gamma(1) = \mathrm{P}\exp\left(-\mathrm{i}\int_\gamma A_m\,\mathrm{d}x^m\right) \ .$$

Let $\gamma = \gamma_0$ be the geodesic connecting x and x':

$$\gamma_0(t) = t\,(x - x') + x' \ .$$

The two-point function

$$I(x,x') \equiv I_{\gamma_0}(x,x')$$

will be called the *parallel displacement propagator*.

DeWitt (1963)

Main properties of the parallel displacement propagator:

(i) gauge transformation law

$$I(x,x') \;\to\; \mathrm{e}^{\mathrm{i}\tau(x)}\, I(x,x')\, \mathrm{e}^{-\mathrm{i}\tau(x')} \;;$$

(ii) boundary condition

$$I(x,x) = \mathbf{1} \;;$$

(iii) master equation

$$(x-x')^a\, \nabla_a\, I(x,x') = (x-x')^a \Big(\partial_a + \mathrm{i}\, A_a(x)\Big)\, I(x,x') = 0 \;.$$

The master equation implies

$$(x-x')^{a_1} \ldots (x-x')^{a_n}\, \nabla_{a_1} \ldots \nabla_{a_n}\, I(x,x') = 0 \;, \qquad n>0 \;,$$

and therefore

$$\nabla_{(a_1} \ldots \nabla_{a_n)}\, I(x,x')\big|_{x=x'} = 0 \;, \qquad n>0 \;.$$

Further property:

$$I(x,x')\, I(x',x) = \mathbf{1} \;.$$

By hitting this identity with $(x-x')^a\, \partial'_a$, and then adding $(x-x')^a\, I(x,x')(\mathrm{i}\, A_a(x') - \mathrm{i}\, A_a(x'))I(x',x) = 0$, we get

$$(x-x')^a\, \nabla'_a\, I(x,x') = (x-x')^a \Big(\partial'_a\, I(x,x') - \mathrm{i}\, I(x,x')\, A_a(x')\Big) \;= 0 \;.$$

Hermitian conjugation:

$$\Big(I(x,x')\Big)^{\dagger} = I(x',x) \;.$$

Covariant Taylor expansion

Barvinsky, Vilkovisky (1985)

Let $\varphi(x)$ be a field transforming in some representation of the gauge group. Then

$$\varphi(x) = I(x,x')\sum_{n=0}^{\infty}\frac{1}{n!}(x-x')^{a_1}\dots(x-x')^{a_n}\,\nabla'_{a_1}\dots\nabla'_{a_n}\varphi(x')\ .$$

Barvinsky, Vilkovisky (1985)

The covariant Taylor expansion implies the following:

Identity (*)

$$\nabla_b I(x,x') = \mathrm{i}\, I(x,x')\sum_{n=1}^{\infty}\frac{n}{(n+1)!}(x-x')^{a_1}\dots(x-x')^{a_n} \times \nabla'_{a_1}\dots\nabla'_{a_{n-1}}F_{a_n\,b}(x')\ ,$$

or equivalently

Identity (**)

$$\nabla_b I(x,x') = -\mathrm{i}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)!}(x-x')^{a_1}\dots(x-x')^{a_n} \times \nabla_{a_1}\dots\nabla_{a_{n-1}}F_{a_n\,b}(x)\,I(x,x')\ .$$

Avramidi (1990,2000)

Derivation is given in the Appendices.

Fock-Schwinger gauge

Let us fix some space-time point x' and consider the following gauge transformation:

$$\mathrm{e}^{\mathrm{i}\tau(x)} = I(x', x) \;, \qquad \mathrm{e}^{\mathrm{i}\tau(x')} = \mathbf{1} \;.$$

Applying this gauge transformation to $I(x, x')$,

$$I(x, x') \;\to\; \mathrm{e}^{\mathrm{i}\tau(x)}\, I(x, x')\, \mathrm{e}^{-\mathrm{i}\tau(x')} \;,$$

the result is

$$I(x, x') = \mathbf{1} \;,$$

which is equivalent, due to $(x - x')^a\, \nabla_a\, I(x, x') = 0$, to the Fock-Schwinger gauge

$$(x - x')^m\, A_m(x) = 0 \;.$$

Fock (1937)

Schwinger (1951,1973)

In the Fock-Schwinger gauge, the identity $(*)$ becomes

$$A_b(x) = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}\, (x - x')^{a_1} \ldots (x - x')^{a_{n-1}} (x - x')^{a_n} \times \nabla'_{a_1} \ldots \nabla'_{a_{n-1}} F_{a_n\, b}(x') \;.$$

Shifman (1980)

Thus, all coefficients in the Taylor expansion of $A(x)$ acquire a geometric meaning.

Proper-time representation:

$$G(x,x') = \mathrm{i}\int_0^\infty \mathrm{d}s\, K(x,x'|s)\ ,$$

where the so-called `heat kernel` $K(x,x'|s)$ is formally given by

$$K(x,x'|s) = \mathrm{e}^{\mathrm{i}s\,(\Delta+\mathrm{i}\,\varepsilon)}\,\delta^d(x-x')\,\mathbf{1}\ , \qquad \varepsilon \to +0\ ,$$

and possesses the gauge transformation

$$K(x,x'|s) \ \longrightarrow\ \mathrm{e}^{\mathrm{i}\tau(x)}\,K(x,x'|s)\,\mathrm{e}^{-\mathrm{i}\tau(x')}\ .$$

`Covariant momentum representation:`

$$\delta^d(x-x')\,\mathbf{1} = \delta^d(x-x')\,I(x,x') = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}\,k.(x-x')}\,I(x,x')\ ,$$

$$\mathrm{e}^{\mathrm{i}\,k.(x-x')}\,I(x,x') \ \longrightarrow\ \mathrm{e}^{\mathrm{i}\tau(x)}\,\Big\{\mathrm{e}^{\mathrm{i}\,k.(x-x')}\,I(x,x')\Big\}\,\mathrm{e}^{-\mathrm{i}\tau(x')}\ .$$

The heat kernel takes the form

$$K(x,x'|s) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}\,k.(x-x')}\,\mathrm{e}^{\mathrm{i}s[(\nabla+\mathrm{i}k)^2-\mathcal{U}]}\,I(x,x')$$

$$= \frac{1}{(4\pi^2 s)^{d/2}}\int \mathrm{d}^d k\,\mathrm{e}^{-\mathrm{i}k^2+\mathrm{i}\,s^{-1/2}\,k.(x-x')}\,\mathrm{e}^{[\mathrm{i}s\nabla^2-2s^{1/2}k.\nabla-\mathrm{i}\,s\,\mathcal{U}]}\,I(x,x')\ .$$

The second exponential should be expanded in a Taylor series. Whenever a covariant derivative ∇_b from this series hits $I(x,x')$, we apply the identity $(*)$. Given a product $\mathcal{U}(x)\,I(x,x')$, we represent it as

$$\mathcal{U}(x)\,I(x,x') = I(x,x')\sum_{n=0}^\infty \frac{1}{n!}\,(x-x')^{a_1}\dots(x-x')^{a_n}\,\nabla'_{a_1}\dots\nabla'_{a_n}\mathcal{U}(x')\ .$$

A generic term in the Taylor expansion involves a Gaussian moment of the form

$$\langle k^{a_1}\dots k^{a_n}\rangle \equiv \frac{1}{(4\pi^2 s)^{d/2}}\int \mathrm{d}^d k\, \mathrm{e}^{-\mathrm{i}k^2+\mathrm{i}\, s^{-1/2}\, k.(x-x')}\, s^{1/2}k^{a_1}\dots s^{1/2}k^{a_n}\ ,$$

where each k^{a_i} comes together with an s-independent factor of ∇_{a_i}; there also occur insertions of $s\nabla^2$ and $s\mathcal{U}$. To compute the moments, introduce a generating function $Z(J)$,

$$Z(J) = \frac{1}{(4\pi^2 s)^{d/2}}\int \mathrm{d}^d k\, \mathrm{e}^{-\mathrm{i}k^2+\mathrm{i}\, s^{-1/2}\, k.(x-x')+s^{1/2}\, J.k}\ ,$$

$$\langle k^{a_1}\dots k^{a_n}\rangle = \frac{\partial^n}{\partial J_{a_1}\dots\partial J_{a_n}} Z(J)\big|_{J=0}\ ,$$

$$Z(J) = \frac{\mathrm{i}}{(4\pi \mathrm{i} s)^{d/2}}\, \mathrm{e}^{\mathrm{i}(x-x')^2/4s}\, \mathrm{e}^{-\mathrm{i}sJ^2/4+J.(x-x')/2}\ .$$

As a result, the heat kernel takes the Schwinger-DeWitt form:

$$K(x,x'|s) = \frac{\mathrm{i}}{(4\pi \mathrm{i} s)^{d/2}}\, \mathrm{e}^{\mathrm{i}(x-x')^2/4s}\sum_{n=0}^{\infty} a_n(x,x')\,(\mathrm{i}s)^n\ ,$$

where

$$a_0(x,x') = \sum_{p=0}^{\infty}\frac{1}{p!}(x'-x)^{m_1}\dots(x'-x)^{m_p}\,\nabla_{m_1}\dots\nabla_{m_p} I(x,x') = I(x,x')\ .$$

The Schwinger-DeWitt coefficients a_n have the form

$$\begin{aligned} a_n(x,x') &= \mathbf{a}_n(F(x),\nabla F(x),\dots,\mathcal{U}(x),\nabla\mathcal{U}(x)\dots;x-x')\, I(x,x') \\ &= I(x,x')\,\mathbf{a}'_n(F(x'),\nabla' F(x'),\dots,\mathcal{U}(x'),\nabla'\mathcal{U}(x')\dots;x-x')\ , \end{aligned}$$

where the functions $\mathbf{a}_n$ and $\mathbf{a}'_n$ are straightforward to compute using the scheme described above.

Covariant derivative expansion in SYM theory

$z^m = (x^m, \theta^\mu, \bar\theta_{\dot\mu})$ coordinates of $\mathcal{N} = 1$ superspace.

$D_A = (\partial_a, D_\alpha, \bar D^{\dot\alpha})$ flat superspace covariant derivatives.

`Supersymmetric Cartan 1-forms` $\omega^A = (\omega^a, \omega^\alpha, \bar\omega_{\dot\alpha})$

$$\mathrm{d}z^M\,\partial_M = \omega^A\,D_A\ , \qquad \omega^A = (\mathrm{d}x^a - \mathrm{i}\,\mathrm{d}\theta\sigma^a\bar\theta + \mathrm{i}\,\theta\sigma^a\mathrm{d}\bar\theta, \mathrm{d}\theta^\alpha, \mathrm{d}\bar\theta_{\dot\alpha})\ .$$

Let $z^M(t) = (z - z')^M\,t + z'^M$ be the straight line connecting two points z and z' in superspace, $z^M(0) = z'^M$ and $z^M(1) = z^M$. We then have $\dot z^M\,\partial_M = \zeta^A\,D_A$, where the two-point function $\zeta^A \equiv \zeta^A(z,z') = -\zeta^A(z',z)$ is

$$\zeta^A = \begin{cases} \rho^a = (x - x')^a - \mathrm{i}(\theta - \theta')\sigma^a\bar\theta' + \mathrm{i}\theta'\sigma^a(\bar\theta - \bar\theta')\ , \\ \zeta^\alpha = (\theta - \theta')^\alpha\ , \\ \bar\zeta_{\dot\alpha} = (\bar\theta - \bar\theta')_{\dot\alpha}\ . \end{cases}$$

The **parallel displacement propagator** along the straight line, $I(z,z')$, is specified by the requirements:

(i) the gauge transformation law

$$I(z,z')\ \to\ \mathrm{e}^{\mathrm{i}\tau(z)}\,I(z,z')\,\mathrm{e}^{-\mathrm{i}\tau(z')}\ ;$$

(ii) the equation

$$\zeta^A\mathcal{D}_A\,I(z,z') = \zeta^A\Big(D_A + \mathrm{i}\,\Gamma_A(z)\Big)I(z,z') = 0\ ;$$

(iii) the boundary condition

$$I(z,z) = \mathbf{1}\ .$$

`Consequences:`

$$I(z,z')\,I(z',z) = \mathbf{1}\ .$$

We also have

$$\zeta^A \mathcal{D}'_A\, I(z,z') = \zeta^A \Big(D'_A\, I(z,z') - \mathrm{i}\, I(z,z')\, \Gamma_A(z') \Big) = 0\ .$$

Further, using the identity

$$\zeta^B\, D_B \zeta^A = \zeta^A\ ,$$

from the master equation one deduces

$$\zeta^{A_n} \dots \zeta^{A_1}\, \mathcal{D}_{A_1} \dots \mathcal{D}_{A_n}\, I(z,z') = 0\ .$$

The latter leads to

$$\mathcal{D}_{(A_1} \dots \mathcal{D}_{A_n\}}\, I(z,z')\big|_{z=z'} = 0\ , \qquad n \geq 1\ ,$$

where $(\dots\}$ means graded symmetrization of n indices (with a factor of $1/n!$).

`Covariant Taylor expansion`

Let $\Psi(z)$ be a superfield transforming in some representation of the gauge group,

$$\Psi(z)\ \to\ \mathrm{e}^{\mathrm{i}\tau(z)}\, \Psi(z)\ .$$

Then

$$\Psi(z) = I(z,z') \sum_{n=0}^{\infty} \frac{1}{n!}\, \zeta^{A_n} \dots \zeta^{A_1}\, \mathcal{D}'_{A_1} \dots \mathcal{D}'_{A_n}\, \Psi(z')\ .$$

The covariant Taylor expansion implies

Identity ($\star$)

$$\mathcal{D}_B I(z,z') = \mathrm{i}\, I(z,z') \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \Big\{ n\, \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}'_{A_1} \dots \mathcal{D}'_{A_{n-1}} \mathcal{F}_{A_n\, B}(z') + \frac{1}{2}(n-1)\, \zeta^{A_n} T_{A_n\, B}{}^{C}\, \zeta^{A_{n-1}} \dots \zeta^{A_1} \mathcal{D}'_{A_1} \dots \mathcal{D}'_{A_{n-2}} \mathcal{F}_{A_{n-1}\, C}(z') \Big\} ~,$$

or equivalently

Identity ($\star\star$)

$$\mathcal{D}_B I(z,z') = \mathrm{i} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \Big\{ - \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}_{A_1} \dots \mathcal{D}_{A_{n-1}} \mathcal{F}_{A_n\, B}(z) + \frac{1}{2}(n-1)\, \zeta^{A_n} T_{A_n\, B}{}^{C}\, \zeta^{A_{n-1}} \dots \zeta^{A_1} \mathcal{D}_{A_1} \dots \mathcal{D}_{A_{n-2}} \mathcal{F}_{A_{n-1}\, C}(z) \Big\} \times I(z,z') ~.$$

SMK, McArthur (2003)

Supersymmetric Fock-Schwinger gauge

$$I(z,z') = \mathbf{1} \quad \Longleftrightarrow \quad \zeta^A\, \Gamma_A(z) = 0 ~.$$

Orndorf (1986)

In this gauge

$$\Gamma_B(z) = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \{ n\, \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}'_{A_1} \dots \mathcal{D}'_{A_{n-1}} \mathcal{F}_{A_n\, B}(z') + \frac{1}{2}(n-1)\, \zeta^{A_n} T_{A_n\, B}{}^{C}\, \zeta^{A_{n-1}} \dots \zeta^{A_1} \mathcal{D}'_{A_1} \dots \mathcal{D}'_{A_{n-2}} \mathcal{F}_{A_{n-1}\, C}(z') \} ~.$$

We are finally prepared to study the superspace Green's function introduced at the beginning of this lecture.

$$(\Box_{\rm v} - m^2)\, G(z,z') = -\mathbf{1}\, \delta^8(z-z')\ .$$

Introduce the proper-time representation for G

$$G(z,z') = {\rm i} \int_0^\infty {\rm d}s\, {\rm e}^{-{\rm i}s\,(m^2 - {\rm i}\,\varepsilon)}\, K(z,z'|s)\ , \qquad \varepsilon \longrightarrow +0\ ,$$

where the `heat kernel` $K(z,z'|s)$ has the formal representation

$$K(z,z'|s) = {\rm e}^{{\rm i}s\,\Box_{\rm v}}\, \delta^8(z-z')\, \mathbf{1}\ ,$$

and possesses the gauge transformation

$$K(z,z'|s) \ \rightarrow\ {\rm e}^{{\rm i}\tau(z)}\, K(z,z'|s)\, {\rm e}^{-{\rm i}\tau(z')}\ .$$

Momentum representation for the superspace delta function:

$$\delta^8(z-z') = \int \frac{{\rm d}^4k}{(2\pi)^4}\, {\rm e}^{{\rm i}k^a(x-x')_a}\, \zeta^2\bar{\zeta}^2 = \int \frac{{\rm d}^4k}{(2\pi)^4}\, {\rm e}^{{\rm i}k^a\rho_a}\, \zeta^2\bar{\zeta}^2$$
$$= \frac{1}{\pi^4}\int {\rm d}^4k \int {\rm d}^2\kappa \int {\rm d}^2\bar{\kappa}\ {\rm e}^{{\rm i}\,[k^a\rho_a + \kappa^\alpha\zeta_\alpha + \bar{\kappa}_{\dot\alpha}\bar{\zeta}^{\dot\alpha}]}\ .$$

Covariant momentum representation:

$$\delta^8(z-z')\,\mathbf{1} = \int \frac{{\rm d}^4k}{(2\pi)^4}\, {\rm e}^{{\rm i}k^a\rho_a}\, \zeta^2\bar{\zeta}^2\, I(z,z')$$
$$= \frac{1}{\pi^4}\int {\rm d}^4k \int {\rm d}^2\kappa \int {\rm d}^2\bar{\kappa}\ {\rm e}^{{\rm i}\,[k^a\rho_a + \kappa^\alpha\zeta_\alpha + \bar{\kappa}_{\dot\alpha}\bar{\zeta}^{\dot\alpha}]}\, I(z,z')\ .$$

The heat kernel can now be represented as follows:

$$K(z,z'|s) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}\, k^a \rho_a}\, \mathrm{e}^{\mathrm{i}s[(\mathcal{D}+\mathrm{i}k)^2 - \mathcal{W}^\alpha \mathcal{D}_\alpha + \bar{\mathcal{W}}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha}]} \times \zeta^2\, \bar\zeta^2\, I(z,z')\ .$$

The covariant derivative expansion for $K(z,z'|s)$ follows from this representation, in complete analogy with the non-supersymmetric case.

Evaluation of the heat kernel obtained can be carried out in a manner almost identical to that outlined for the non-supersymmetric case. The result is the following asymptotic expansion:

$$K(z,z'|s) = -\frac{\mathrm{i}}{(4\pi s)^2}\, \mathrm{e}^{\mathrm{i}\zeta^a \zeta_a/4s} \sum_{n=0}^{\infty} a_n(z,z')\, (\mathrm{i}s)^n\ ,$$

where

$$\begin{aligned} a_0(z,z') &= \sum_{n=0}^{\infty} \frac{1}{n!}\, (-1)^n\, \rho^{a_n} \dots \rho^{a_1}\, \mathcal{D}_{a_1} \dots \mathcal{D}_{a_n}\ \delta^4(\theta-\theta')\, I(z,z') \\ &= \delta^4(\theta-\theta') \sum_{n=0}^{\infty} \frac{1}{n!}\, (-1)^n\, \rho^{a_n} \dots \rho^{a_1}\, \mathcal{D}_{a_1} \dots \mathcal{D}_{a_n}\ I(z,z') \\ &= \delta^4(\theta-\theta') \sum_{n=0}^{\infty} \frac{1}{n!}\, (-1)^n\, \zeta^{A_n} \dots \zeta^{A_1}\, \mathcal{D}_{A_1} \dots \mathcal{D}_{A_n}\ I(z,z') \\ &= \delta^4(\theta-\theta')\, I(z,z') = \zeta^2\, \bar\zeta^2\, I(z,z')\ . \end{aligned}$$

Append. A: Derivation of covariant Taylor expansion

Consider the straight line connecting two points z and z'.

$$z^M(t) = (z - z')^M \, t + z'^M \ , \qquad z(0) = z' \ , \quad z(1) = z \ ,$$
$$\dot{z}^M \, \partial_M = \zeta^A \, D_A \ , \qquad \frac{\mathrm{d}}{\mathrm{d}t} \, \zeta^A = 0 \ .$$

Given a `gauge invariant` superfield $U(z)$, for $U(t) = U(z(t))$ we have

$$\frac{\mathrm{d}^n U}{\mathrm{d}t^n} = \zeta^{A_n} \ldots \zeta^{A_1} \, D_{A_1} \ldots D_{A_n} \, U \ ,$$

since $\dot{\zeta}^A = 0$. This leads to a *supersymmetric Taylor series*

$$U(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \, \zeta^{A_n} \ldots \zeta^{A_1} \, D'_{A_1} \ldots D'_{A_n} \, U(z') \ .$$

Now, let $\Psi(z)$ be a superfield transforming in some representation of the gauge group. Then $U(z) \equiv I(z', z) \, \Psi(z)$ is gauge invariant with respect to z, and therefore we are in a position to apply the supersymmetric Taylor expansion.

$$\Psi(z) = I(z, z') \, \sum_{n=0}^{\infty} \frac{1}{n!} \, \zeta^{A_n} \ldots \zeta^{A_1} \, \mathcal{D}_{A_1} \ldots \mathcal{D}_{A_n} \left(I(z', w) \, \Psi(w) \right) \big|_{w=z'} \ .$$

This is equivalent to the covariant Taylor series, since

$$\mathcal{D}_{(A_1} \ldots \mathcal{D}_{A_n\}} \, I(z', z) \big|_{z=z'} = 0 \ , \qquad n \geq 1 \ .$$

Append. B: Derivation of identity ($\star$)

Apply the covariant Taylor expansion to $\mathcal{D}_B I(z,z')$ considered as a superfield at z,

$$\mathcal{D}_B I(z,z') = I(z,z') \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}_{A_1} \dots \mathcal{D}_{A_n} \mathcal{D}_B I(w,z')|_{w=z'} \ .$$

We start with an obvious identity

$$(n+1)\, \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}_{(A_1} \dots \mathcal{D}_{A_n} \mathcal{D}_{B\}} = \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}_{A_1} \dots \mathcal{D}_{A_n} \mathcal{D}_B \\ + \sum_{i=1}^{n} (-1)^{B(A_i+\dots+A_n)} \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}_{A_1} \dots \mathcal{D}_{A_{i-1}} \mathcal{D}_B \mathcal{D}_{A_i} \dots \mathcal{D}_{A_n} \ ,$$

and make use of the property of

$$\mathcal{D}_{(A_1} \dots \mathcal{D}_{A_n} \mathcal{D}_{B\}}\, I(z,z')|_{z=z'} = 0 \ .$$

We thus have

$$0 = \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}_{A_1} \dots \mathcal{D}_{A_n} \mathcal{D}_B\, I(z,z')|_{z=z'} \qquad (\star\star\star) \\ + \sum_{i=1}^{n} (-1)^{B(A_i+\dots+A_n)} \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}_{A_1} \dots \mathcal{D}_{A_{i-1}} \mathcal{D}_B \mathcal{D}_{A_i} \dots \mathcal{D}_{A_n}\, I(z,z')|_{z=z'} \ .$$

The next step is to represent

$$(-1)^{B(A_i+\dots+A_n)} \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}_{A_1} \dots \mathcal{D}_{A_{i-1}} \mathcal{D}_B \mathcal{D}_{A_i} \dots \mathcal{D}_{A_n} \\ = -\, (-1)^{B(A_{i+1}+\dots+A_n)} \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}_{A_1} \dots \mathcal{D}_{A_{i-1}} [\mathcal{D}_{A_i}, \mathcal{D}_B\} \mathcal{D}_{A_{i+1}} \dots \mathcal{D}_{A_n} \\ + (-1)^{B(A_{i+1}+\dots+A_n)} \zeta^{A_n} \dots \zeta^{A_1} \mathcal{D}_{A_1} \dots \mathcal{D}_{A_i} \mathcal{D}_B \mathcal{D}_{A_{i+1}} \dots \mathcal{D}_{A_n}$$

and make use of the covariant derivative algebra,

$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^{C}\, \mathcal{D}_C + \mathrm{i}\, \mathcal{F}_{AB} \ ,$$

along with the observation

$$(-1)^{B(A_{i+1}+\ldots+A_n)}\zeta^{A_n}\ldots\zeta^{A_1}\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_{i-1}}\mathcal{F}_{A_i\,B}\mathcal{D}_{A_{i+1}}\ldots\mathcal{D}_{A_n}\,I(z,z')|_{z=z'}$$
$$=\begin{cases} 0\ , & i<n\ ; \\ \zeta^{A_n}\ldots\zeta^{A_1}\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_{n-1}}\,\mathcal{F}_{A_n\,B}\ , & i=n\ . \end{cases}$$

Repeating this procedure, each contribution to the second terms in $(\star\star\star)$ can be reduced to the first term plus additional terms involving graded commutators of covariant derivatives. Since the torsion $T_{AB}{}^C$ is `constant`, we then obtain

$$\begin{aligned}
&(n+1)\,\zeta^{A_n}\ldots\zeta^{A_1}\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_n}\mathcal{D}_B I(z,z')|_{z=z'} \\
&=\sum_{i=1}^{n}(-1)^{C(A_{i+1}+\ldots+A_n)}\zeta^{A_i}T_{A_i\,B}{}^C\zeta^{A_n}\ldots\underbrace{1}_{i}\ldots\zeta^{A_1} \\
&\qquad\times\mathcal{D}_{A_1}\ldots\underbrace{\mathcal{D}_C}_{i}\ldots\mathcal{D}_{A_n}\,I(z,z')|_{z=z'} \\
&\quad+n\mathrm{i}\,\zeta^{A_n}\ldots\zeta^{A_1}\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_{n-1}}\,\mathcal{F}_{A_n\,B}\ .
\end{aligned}$$

For the first term in the right hand side, we can again apply the previous procedure, and this now simplifies since

$$T_{AB}{}^C\,[\mathcal{D}_C,\mathcal{D}_D\} = (-1)^C\,T_{AB}{}^C\,[\mathcal{D}_C,\mathcal{D}_D\} = \mathrm{i}\,T_{AB}{}^C\,\mathcal{F}_{CD}\ .$$

After some algebra, one then arrives at $(n>0)$

$$\begin{aligned}
&(n+1)\,\zeta^{A_n}\ldots\zeta^{A_1}\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_n}\mathcal{D}_B I(z,z')|_{z=z'} \\
&=\mathrm{i}\,n\,\zeta^{A_n}\ldots\zeta^{A_1}\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_{n-1}}\mathcal{F}_{A_n\,B} \\
&\quad+\frac{\mathrm{i}}{2}\,(n-1)\,\zeta^{A_n}T_{A_n\,B}{}^C\,\zeta^{A_{n-1}}\ldots\zeta^{A_1}\mathcal{D}_{A_1}\ldots\mathcal{D}_{A_{n-2}}\mathcal{F}_{A_{n-1}\,C}\ .
\end{aligned}$$