Condensed Matter Theory — problem set 8

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Problem 21: Polarized electron gas

Consider a polarized electron gas with N_+ spin-up and N_- spin-down electrons in a uniform positive charge background.

(a) Find the ground-state energy to first order in the interaction potential (Hartree-Fock) as a function of $N = N_+ + N_-$ and the polarization $\zeta = (N_+ - N_-)/N$.

(b) Prove that the ferromagnetic state ($\zeta = 1$) represents a lower energy than the unpolarized (paramagnetic) state ($\zeta = 0$) if $r_s > \left(\frac{2\pi}{5}\right)\left(\frac{9\pi}{45}\right)^{1/3}(2^{1/3}+1) \approx 5.45$.

(c) Show that $\frac{\partial^2(E/N)}{\partial\zeta^2}$ at $\zeta = 0$ becomes negative for $r_s > \left(\frac{3\pi^2}{2}\right)^{2/3} \approx 6.03$.

(d) What happens to the paramagnetic state for $5.45 < r_s < 6.03$?

Notation: We define $r_s = r_0/a_0$ with $a_0 = \hbar^2/(me^2m)$ the Bohr radius and r_0 the average particle spacing, as in the lecture.

Problem 22: Random-phase approximation

The leading-order polarization function of the uniform Fermi gas is given by

$$\Pi(\boldsymbol{q}, i\omega_m) = \frac{1}{\beta V}\sum_{\boldsymbol{k},n,\sigma} \mathcal{G}_0(\boldsymbol{k}, i\epsilon_n)\,\mathcal{G}_0(\boldsymbol{k}+\boldsymbol{q}, i\epsilon_n + i\omega_m) \tag{1}$$

with fermionic and bosonic Matsubara frequencies ϵ_n and ω_m, respectively. The fermionic thermal Green function is $\mathcal{G}_0^{-1}(\boldsymbol{k}, i\epsilon_n) = i\epsilon_n - \xi_{\boldsymbol{k}}$ with $\xi_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}} - \mu$ and $\varepsilon_{\boldsymbol{k}} = k^2/2m$.

(a) Evaluate the Matsubara sum and derive the expression

$$\Pi(\boldsymbol{q}, i\omega_m) = \sum_\sigma \int \frac{d^3k}{(2\pi)^3}\,\frac{f(\xi_{\boldsymbol{k}}) - f(\xi_{\boldsymbol{k}+\boldsymbol{q}})}{i\omega_m - \varepsilon_{\boldsymbol{k}+\boldsymbol{q}} + \varepsilon_{\boldsymbol{k}}} \tag{2}$$

where $f(\varepsilon)$ is the Fermi function.

(b) Compute the momentum sum in the static case $i\omega_m = 0$ at $T = 0$, using the function $F(x)$ derived in the lecture for the Fock self-energy.