

Quantum Field Theory I

Assignment Week 5

We are aware that this exercise is (very) long. But remember you don't have to spend all your week on them and you don't have to be able to solve everything. An important skill is to learn to spend your time smart and efficiently on what matters/interests you more. So here is a rough priority list:

- i) Exercise 1, questions c) and f)
- ii) Exercise 2, questions e) and d)

Classroom Exercise 1: Oscillating = cancelling

*Motivation: In the lecture, we saw that transition amplitudes can be formulated as certain types of functional integrals, called **path integrals**. The dominant contributions to these amplitudes comes from classical paths/trajectories in quantum mechanics. This exercise aims to provide some intuition behind the relation of the stationary action and the path integral formulation.*

- A) **The oscillatory nature of the path integral.** We consider the path integral of a one dimensional harmonic oscillator with mass m and frequency ω ,

$$Z = \int Dq \Big|_{q(0)=q_0}^{q(T)=q_f} \exp \left[\frac{i}{\hbar} \int_0^T dt \left(\frac{m}{2} \dot{q}^2 - \frac{m\omega^2}{2} q^2 \right) \right]. \quad (1.1)$$

By definition the path integral sums over all paths satisfying the chosen boundary conditions. Therefore, it also sums non-classical paths, i.e. paths that do not solve the equations of motion. We will showcase though that the classical path/configuration q_{cl} dominates this integral. To that end, we consider non-classical deformations around the classical path written as:

$$q(t) = q_{cl}(t) + \varepsilon f(t) \quad (1.2)$$

where $q_{cl}(t)$ is the classical path satisfying the boundary conditions in the path integral and $f(t)$ a non-classical deformation satisfying homogeneous boundary conditions.

- i) Use Eq. (1.2) to expand the action to second order in ε . Show that

$$S[x] = S_{cl} + \varepsilon^2 S_2 + O(\varepsilon^3), \quad (1.3)$$

$$\text{with } S_2 = \frac{m}{2} \int_0^T dt (\dot{f}(t)^2 - \omega^2 f(t)^2).$$

- ii) The path integral weight becomes

$$e^{iS[x]} = e^{iS_{cl}} e^{i\varepsilon^2 S_2} = e^{iS_{cl}} (\cos(\varepsilon^2 S_2) + i \sin(\varepsilon^2 S_2)). \quad (1.4)$$

How does the ε -dependence behave for large ε ? You can also plot $\cos(\varepsilon^2 S_2)$ or $\sin(\varepsilon^2 S_2)$. Averaging (or summing) all possible large deformations can be seen as integrating over ε leaving the action S_2 as a free parameter. In the overall sum, the dominant contribution

comes from the regions of small ε . To see this use your favourite computational tool and plot the function

$$I(\epsilon) = \int_{-\epsilon}^{\epsilon} \cos(\varepsilon^2) d\varepsilon . \quad (1.5)$$

Note that the above intuition we gained is independent of the choice of action! What happens to the path integral in the limit $\hbar \rightarrow 0$?

B) **A useful identity.** Let C be a finite dimensional, symmetric, and non-degenerate matrix, i.e. its determinant is non-zero. Show that:

$$\det(e^C) = e^{\text{Tr}(C)} . \quad (1.6)$$

Hint: Apply Jacobi's formula which for any invertible matrix takes the form

$$\frac{d}{dt} \det(A(t)) = \det(A(t)) \text{Tr} \left[A^{-1}(t) \frac{d}{dt} A(t) \right] , \quad (1.7)$$

and solve the partial differential equation for $\det(A(t))$ with $A(t) = e^{tC}$.

Keep in mind, that the condition of non-degeneracy is crucial for using Jacobi's formula, otherwise the matrix is not invertible. This will be important later in the course! If the logarithm of a matrix A exists and is unique then Eq. (1.6) leads to

$$\det(A) = e^{\text{Tr}(\log A)} . \quad (1.8)$$

Exercise 1: Functionals & Functional Derivatives

Motivation: In this exercise, we will explore the notion of functionals and functional derivatives. Functional derivatives are very important for physics and Quantum Field Theory in particular. Important objects like the propagator (or more generally correlation functions as you will discuss later in the lecture) can be recasted as functional derivatives of certain “generating” functionals. So here, we clarify and practice with these notions, which you actually have been somewhat familiar with already.

As you know functions are maps from continuous variables in some domain, for example the coordinates of spacetime points, to some number (real or complex). A **functional** $\mathcal{F}$ however is a map that maps functions, in some space onto a number. In Quantum Field Theory, one is usually interested in functionals of the quantum fields. We will focus on the functionals $\mathcal{F}[f]$ that can be written as an integral of f as^a

$$\mathcal{F}[f] = \int g(x) f(x) dx , \quad (2.9)$$

for some function (or generalized function) $g(x)$. A probably trivial example, is the functional that takes functions and evaluates them at some point x_0 , and it can be written as

$$\mathcal{F}[f] = f(x_0) = \int \delta(x - x_0) f(x) dx , \quad (2.10)$$

or in a more suggestive notation

$$\mathcal{F}[f] = f(y) = \int \delta(x - y) f(x) dx , \quad (2.11)$$

which means that from the view of the functional the variable y is a parameter rather than a variable. A more non-trivial example is the action functional of the free real scalar field

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 \right) \quad (2.12)$$

Now we would like to define a **functional derivative** that extends normal derivatives on functions. To that end we consider a general infinitesimal variations of the argument function $f(x) + \epsilon \eta(x)$ where $\eta(x)$ is an arbitrary function. We define the functional derivative $\frac{\delta \mathcal{F}[f]}{\delta f(y)}$ through the relation

$$\int \frac{\delta \mathcal{F}[f]}{\delta f(y)} \eta(y) dy = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{F}[f + \epsilon \eta] - \mathcal{F}[f]}{\epsilon} \quad (2.13)$$

Therefore, to calculate such a derivative one need to calculate the variation $\delta \mathcal{F}[f] = \mathcal{F}[f + \epsilon \eta] - \mathcal{F}[f]$ and write it as an integral linear in the variation η . Then one identifies the functional derivative through Eq. (2.13). Let's do an example:

- a) Calculate the functional derivative of the following functional

$$\mathcal{F}[f] = f(x) = \int \delta(x - z) f(z) dz , \quad (2.14)$$

by determining the right-hand side of Eq. (2.13). You already did this in the lecture but it is good to repeat it.

- b) Calculate the functional derivative of the following functional

$$\mathcal{F}[f] = \int f(x)^p dx , p \in \mathbb{Q} \quad (2.15)$$

by determining the right-hand side of Eq. (2.13). You should find that

$$\frac{\delta \mathcal{F}[f]}{\delta f(x)} = p f(x)^{p-1} . \quad (2.16)$$

Hint: Use the binomial theorem $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$.

In physics, we make our life easier by choosing the arbitrary function $\eta(x)$ to be the δ -function, which eliminates the integral of the left-hand side of Eq.(2.13), leading to the formal calculational rule

$$\frac{\delta \mathcal{F}[f]}{\delta f(x_0)} = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{F}[f(x) + \epsilon \delta(x - x_0)] - \mathcal{F}[f(x)]}{\epsilon} \quad (2.17)$$

For all our purposes the prescription of Eq. (2.17) is equivalent to the definition (2.13) while formally, Eq. (2.17) is understood as using a representation of the δ -function as a sequence of smooth and regular functions.

Eq. (2.17) looks suspiciously similar to the definition of a normal derivatives of polynomials. In fact, the familiar derivative rules, such as the chain rule or the product rule, extend to functional derivatives.

Chain rule. Consider a functional $\mathcal{F}$ which depends on some function $G(y)$, which itself is a functional of $f(x)$, we denote this as $G[f](y)$. Then, $\mathcal{F}$ is also a functional of $f(x)$ and its functional derivative is,

$$\frac{\delta \mathcal{F}[G[f]]}{\delta f(x)} = \int \frac{\delta \mathcal{F}[f]}{\delta G(y)} \frac{\delta G[f](y)}{\delta f(x)} dy . \quad (2.18)$$

- c) Recalculate the functional derivatives of the question a) and b) but using rule (2.17), as well as the second functional derivative $\frac{\delta^2 \mathcal{F}}{\delta f(x) \delta f(y)}$ of the functional

$$\mathcal{F}_w[f] = \int dx_1 \int dx_2 [f(x_1) w(x_1, x_2) f(x_2)] \quad (2.19)$$

- d) Given a function $g(x)$, show that the functional derivative with respect to $f(x)$ of the functionals

$$\mathcal{F}_1[f] = \int g(f(y)) dy , \quad \mathcal{F}_2[f] = \int g(f'(y)) dy \quad (2.20)$$

are respectively

$$\frac{\delta \mathcal{F}_1[f]}{\delta f(x)} = g'(f(x)) , \quad \frac{\delta \mathcal{F}_2[f]}{\delta f(x)} = -\frac{d}{dx} \left(\frac{dg(f')}{df'} \right) \quad (2.21)$$

Hint: For the second you will need to perform a partial integration. Assume that you can neglect the boundary term if the point x is a point inside the domain of the y -integral.

Recall now Hamilton's principle in classical mechanics. It states that the physical path of a free particle minimizes the action functional. In the language of functional derivatives this statement is formulated as finding the roots of the functional derivative of the action. The equations of motion (Euler-Lagrange equations) are compactly written as

$$\frac{\delta S[q, \dot{q}]}{\delta q(t)} = 0 . \quad (2.22)$$

- f) Calculate the functional derivative of the action functional of the harmonic oscillator

$$S[x, \dot{x}] = \int dt \left(\frac{m}{2} (\dot{x}(t))^2 - \frac{\omega^2}{2} x(t)^2 \right) \quad (2.23)$$

^aNot all functionals can be written this way. It depends on the space where the functions f live, e.g. compactly supported functions satisfy this.

Exercise 2: Gaussian Integrals

Motivation: Here we review the calculation of one finite-dimensional Gaussian integrals. The results of these integrals are formally extended to infinite dimensions. These infinite dimensional integrals are essential to understand the path-integral formulation of quantum field theory. In fact, the path integral of the free scalar quantum field theory is exactly such an infinite dimensional integral.

You saw in the lecture, that the transition amplitude of a particle to propagate from one point to another is given by an integral over all possible paths. The paths are summed with a weight factor $\exp[\frac{i}{\hbar}S]$ where S is the classical action. For the scalar field theory we have that:

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \right) . \quad (3.24)$$

This action is quadratic with respect to the field variables and hence the path integral written as

$$\int \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi, \partial_\mu \phi]} , \quad (3.25)$$

is formally an integral of Gaussian type that is infinite dimensional^a, i.e. $\mathcal{D}\phi \propto \lim_{N \rightarrow \infty} \prod_{i=1}^N \phi(x_i)$, for some points x_i that serve as a discretization of the continuous spacetime^b.

We start from the one-dimensional Gaussian integrals.

- a) Prove the following identity for a constant $a \in \mathbb{C}$ with $\Re(a) > 0$:

$$I(a) = \int_{-\infty}^{+\infty} e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}} \quad (3.26)$$

Hint: Calculate $I(a)^2$ by changing from cartesian to polar coordinates.

- b) Using the above result, show the identity

$$I(a, b) = \int_{-\infty}^{+\infty} e^{-\frac{a}{2}x^2 + bx} = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{b^2}{2a}\right) \quad (3.27)$$

- c) Generalize the above results to Gaussians of complex numbers $z = \frac{1}{\sqrt{2}}(x + iy)$, by showing the following:

$$\int dz dz^* e^{-z^* w z} = \frac{2\pi}{w} , \quad \int dz dz^* e^{-z^* w z + u^* z + z^* u} = \frac{2\pi}{w} \exp(u^* w^{-1} u) , \quad (3.28)$$

where $u, w \in \mathbb{C}$ and $\Re(w) > 0$. Hint: For the first one, write the integral in terms of the real and imaginary part of z and use the previous result for the real case. For the second one, complete the square in the exponent.

Now we move to arbitrary higher (but finite) dimensional integrals.

- d) Let $\mathbf{A}$ be a **real, symmetric, positive definite N -dimensional matrix**^c and real vectors $\vec{x} \in \mathbb{R}^N$. Show the following identity for the higher dimensional real Gaussian integral

$$\int d^N x \exp \left(-\frac{1}{2} \vec{x}^T \cdot \mathbf{A} \cdot \vec{x} \right) = \frac{(2\pi)^{N/2}}{\sqrt{\det \mathbf{A}}} , \quad (3.29)$$

where in the exponent we have used the matrix notation $\vec{x}^T \cdot \mathbf{A} \cdot \vec{x} = \sum_{i,j=1}^N x^i \mathbf{A}_{ij} x^j$. Using now this elementary result show the following:

$$\int d^N x \exp \left(-\frac{1}{2} \vec{x}^T \cdot \mathbf{A} \cdot \vec{x} + \vec{J}^T \vec{x} \right) = \frac{(2\pi)^{N/2}}{\sqrt{\det \mathbf{A}}} e^{\frac{1}{2} \vec{J}^T \mathbf{A}^{-1} \vec{J}} \quad (3.30)$$

Hint: Since the matrix $\mathbf{A}$ is symmetric there is an orthogonal transformation $\mathcal{O}$ such that the matrix $\mathcal{O}^T \mathbf{A} \mathcal{O}$ is diagonal. Therefore, you can do a change of variables in Eq. (3.29) to write the exponential as a product of elementary Gaussian integrals.

- e) Now we are going to showcase that these “Gaussian” techniques work in the field theory setting and particularly for the path integrals of free fields. We recall that for the free real scalar field (ignoring an overall normalization constant) we have

$$\mathcal{Z}[J] = \int D\phi \exp \left[\frac{i}{\hbar} \int d^4 x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + J(x) \phi(x) \right) \right] . \quad (3.31)$$

This integral however has a complex exponent and hence it is formally an oscillatory integral. Therefore, we cannot expect it to converge to any particular value. We bypass this problem by introducing a small imaginary part in action which will act as a damping factor. Eventually we would like to compute,

$$\mathcal{Z}[J] = \int D\phi \exp \left[\frac{i}{\hbar} \int d^4 x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} i\epsilon \phi^2 + J(x) \phi(x) \right) \right] , \quad (3.32)$$

and at the end of all our manipulations we would like to take the limit $\epsilon \rightarrow 0$. This is the field theory analog of the vacuum-to-vacuum amplitude in quantum mechanics (see chapter 4.5 in the script).

Alternatively one can use yet another generalization of Gaussian integrals where the exponent is complex. This can be shown via Cauchy’s theorem for complex integrals for the finite-dimensional case and we will see the proof of this in the solutions.

Here we calculate the integral by expanding the field variable around $\phi_0(x)$ which solves the equations of motion for the action with a source in Eq. (3.32). In short, we do a change of variables $\phi \rightarrow \phi_0 + \phi$ and assume $D\phi \rightarrow D\phi$.

- i) Show that the equation of motion for ϕ_0 in the presence of a source is

$$(\square + m^2 - i\epsilon) \phi_0(x) = J(x) , \quad (3.33)$$

and show that the general solution can be written as

$$\phi_0(x) = \int \Delta_F(x-y) J(y) dy , \quad (3.34)$$

where $\Delta_F(x-y)$ is the Green's function of the equation of motion^d, i.e. it satisfies the following:

$$(\square + m^2 - i\epsilon) \Delta_F(x-y) = -\delta^{(3)}(x-y) \quad (3.35)$$

Hint: Plug the expression (3.34) in the left-hand side of the equation of motion.

ii) Apply the expansion $\phi \rightarrow \phi_0 + \phi$ to bring the action in the form

$$S = -\frac{1}{2} \int \phi (\square + m^2 - i\epsilon) \phi d^4x - \frac{1}{2} \int J(x) \Delta_F(x-y) J(y) d^4x d^4y. \quad (3.36)$$

iii) Show that the generating functional takes the form, with $\hbar = 1$,

$$\begin{aligned} \mathcal{Z}[J] &= \mathcal{N} \int D\phi \exp\left[-\frac{i}{2} \int \phi (\square - m^2 + i\epsilon) \phi d^4x\right] \\ &\quad \times \exp\left[-\frac{i}{2} \int J(x) \Delta_F(x-y) J(y) d^4x d^4y\right] \end{aligned} \quad (3.37)$$

iv) Lastly, calculate the integral over ϕ using the Gaussian techniques. Could you calculate $\mathcal{Z}[J]$ just by extending the rules of Gaussian integration? *Hint: Ignore overall constants, that in the infinite dimensional limit would diverge, by absorbing them onto the normalization constant. Compare with Eq. (3.30).*

f) **(BONUS:** Take two functional derivatives of $\mathcal{Z}[J]$ with respect to $J(x)$. What is the result? What happens if you take three functional derivatives? What about four? Do you see any pattern?

^aI stress the word formal here because such an integral cannot be rigorously defined. Nevertheless, we understand it and manipulate it as a Gaussian integral over the field variables.

^bThe limit of the number of point $N \rightarrow \infty$ corresponds to the continuum limit.

^cBy positive definite we mean that all eigenvalues of the matrix are positive.

^dThis is called the Feynman propagator and it is a central object in quantum field theory. You will see in the lecture that with the Feynman propagator we can compute any correlation function of time-ordered field operators which are the quantities we are ultimately interested in.