Quantum field theory in curved spacetime

Assignment 4 – May 21

Exercise 10: Kerr black hole

Motivation: Real black holes spin. That twist adds a whole new layer of physics. Let's dive in.

A rotating black hole cannot be static and spherically symmetric anymore. Instead it is axially symmetric and stationary, *i. e.* it changes in time but this change is time-translation invariant. This is the resulting metric:

$$\mathrm{d}s^2 = \left(1 - \frac{2GMr}{\rho^2}\right)\mathrm{d}t^2 + \frac{4GMar\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\phi - \frac{\rho^2}{\Delta}\mathrm{d}r^2 \\ - \rho^2\mathrm{d}\theta^2 - \left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]\frac{\sin^2\theta}{\rho^2}\mathrm{d}\phi^2, \tag{10.1}$$

where the constants M and $a = J/GM$ (with the angular momentum J) parametrize the geometry, while $\Delta = r^2 - 2GMr + a^2$ and $\rho^2 = r^2 + a^2\cos^2\theta$.
In the following, we learn some of the key properties of this metric. The required computations can become involved due to the complexity of the metric itself, so it may be advised to use symbolic equation manipulation software like Mathematica or SageMath.

(**a**) Take the limit $a \to 0$. Which geometry do you obtain? Do the same for the limit $M \to 0$. (**Hint:** You may have to apply coordinate transformations after the limit to properly identify the resulting geometry. Another way to characterize a geometry (which may be simpler) is to compute the corresponding curvature tensor. While ordinarily it is necessary to compute curvature invariants, in this case the curvature tensor is sufficient.)

(**b**) Depending on the constant parameters, the metric has zero, one or two horizons. If they exist, compute their radii (say r_+ and r_- for $r_+ > r_-$), and the parameter ranges corresponding to the three cases. Which of these regimes are expected to be physical? (**Hint:** For the Kerr metric, horizons and their properties can be read off the radial part of the metric.)

(**c**) As for the Schwarzschild geometry, the singularities at the horizons are mere coordinate singularities. Introducing a Tortoise coordinate

$$\mathrm{d}r^* = \frac{r^2+a^2}{\Delta}\mathrm{d}r, \tag{10.2}$$

we can transform the time and angular coordinates to obtain

$$\mathrm{d}v = \mathrm{d}t + \mathrm{d}r^*, \qquad \mathrm{d}\tilde{\phi} = \mathrm{d}\phi - \frac{a}{\Delta}\mathrm{d}r. \tag{10.3}$$

Here the transformation $\phi \to \tilde{\phi}$ is required so that the basis vectors ∂_r and ∂_ϕ remain linearly independent close to the horizon. Show that the metric in the resulting set of ingoing Eddington-Finkelstein coordinates $(v, r, \theta, \tilde{\phi})$ is regular at the outer horizon.

(**d**) A horizon-generating Killing vector is a Killing vector which is null on the horizon. Given such a horizon-generating Killing vector χ, we can then define the horizon as the submanifold constrained by the condition $\chi^2 = 0$. The surface gravity κ is defined such that

$$\chi^\nu \nabla_\nu \chi_\mu|_{r=r_+} = \kappa \chi_\nu|_{r=r_+}. \tag{10.4}$$

What's the left-hand side of this equation? What does it mean that the right-hand side is non-zero?

Contrary to the Schwarzschild case, the horizon generating Killing vector for the outer horizon is not simply ∂_t. Instead, it reads $\chi = \partial_t + \Omega_H \partial_\phi$ with $\Omega_H = a/(a^2 + r_+^2)$. Demonstrate that the surface gravity at the outer horizon equals

$$\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)}. \tag{10.5}$$

This amounts to the acceleration required to hover on top of the outer horizon. What does this mean for the vacuum state defined by an observer inertially falling into the hole? (**Hint:** Turn the vector equation into a scalar equation by projecting on a vector. Note that on the horizon $\partial_\phi|_{r=r+} = \partial_{\tilde{\phi}}|_{r=r+}$.)

(**e**) Compute the surface gravity in the limit $a \to GM$. Borrowing an analogy with the third law of thermodynamics, what do we learn about trying to spin up a black hole until it is extremal? Indeed one can prove the answer to this problem in full generality in general relativity – the result is called the third law of black hole mechanics.

(**a**) In the limit $a \to 0$

$$\Delta = r^2 - 2GMr, \qquad \rho^2 = r^2. \tag{10.6}$$

Thus, the metric becomes

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \frac{1}{1-\frac{2GM}{r}}\mathrm{d}r^2 \tag{10.7}$$

$$- r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\phi^2, \tag{10.8}$$

which is the Schwarzschild metric. This is in line with the fact that a represents the angular momentum – a Schwarzschild black hole is the non-rotating version of a Kerr black hole.

In the limit $M \to 0$

$$\Delta = r^2 + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta. \tag{10.9}$$

As a result, the metric reads

$$\begin{aligned} \mathrm{d}s^2 =& \mathrm{d}t^2 - \frac{r^2 + a^2\cos^2\theta}{r^2+a^2}\mathrm{d}r^2 - \left(r^2 + a^2\cos^2\theta\right)\mathrm{d}\theta^2 \\ &- \left[(r^2+a^2)^2 - a^2(r^2+a^2)\sin^2\theta\right]\frac{\sin^2\theta}{r^2+a^2\cos^2\theta}\mathrm{d}\phi^2. \end{aligned} \tag{10.10}$$

This can be simplified because

$$\frac{(r^2+a^2)^2 - a^2(r^2+a^2)\sin^2\theta}{r^2+a^2\cos^2\theta} = (r^2+a^2)\frac{r^2 + a^2(1-\sin^2\theta)}{r^2+a^2\cos^2\theta} = r^2 + a^2. \tag{10.11}$$

Thus, the metric reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\mathrm{d}r^2 - \left(r^2 + a^2\cos^2\theta\right)\mathrm{d}\theta^2 - \left(r^2 + a^2\right)\sin^2\theta\mathrm{d}\phi^2. \tag{10.12}$$

This is just Minkowski space in disguise – if by disguise you mean ellipsoidal coordinates. Applying the coordinate transformation to Cartesian coordinates

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi, \qquad y = \sqrt{r^2 + a^2}\sin\theta\sin\phi \qquad , z = r\cos\theta, \tag{10.13}$$

we obtain

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\vec{x}^2. \tag{10.14}$$

(**b**) At horizons in stationary geometries, the radial component of the metric diverges. This happens when $\Delta = 0$. This quadratic equation has the solutions

$$r_\pm = GM \pm \sqrt{(GM)^2 - a^2}. \tag{10.15}$$

Thus, there are 0,1 or 2 real roots if $a > GM$, $a = GM$ or $a < GM$, respectively.

As there is still a ring-shaped singularity (better: "ringularity") inside the hole, the case $a > GM$ is a naked singularity, which violates the cosmic censorship conjecture and would allow access to arbitrarily-large-curvature regions – an effect we appear not to see in observations.

If $a = GM$, the Kerr black hole is extremal, *i. e.* the two horizons are equal. Compared to the black holes we see in the sky, this is still excessively fast. Actually, there is a theorem that in GR you can not spin up a black hole up to $a = 1$, just as in thermodynamics you can't lower the temperature to 0 with a finite number of steps (chrchrm Hawking effect chrchrm).

The physical regime is thus $a < GM$. All the observed black holes fall into this category.

(**c**) Applying the coordinate transformation given in Eqs. (10.2) and (10.3), after some algebra we obtain the metric

$$\begin{aligned}\mathrm{d}s^2 = &\left(1 - \frac{2GMr}{\rho^2}\right)\mathrm{d}v^2 + \frac{4GMar}{\rho^2}\mathrm{d}v\mathrm{d}\phi - 2\mathrm{d}v\mathrm{d}r + 2a\sin^2\theta\mathrm{d}r\mathrm{d}\phi \\ &- \rho^2\mathrm{d}\theta^2 - \left[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right]\frac{\sin^2\theta}{\rho^2}\mathrm{d}\phi^2\end{aligned} \tag{10.16}$$

As there is no Δ in any denominator, the metric is regular at $\Delta = 0$.

(**d**) The left-hand side of Eq. (10.4) is the left-hand side of the geodesic equation. Thus, the right-hand side being non-zero captures the acceleration required to stay put, *i. e.* hover over the horizon.

Killing vectors satisfy the Killing equation

$$\nabla_{(\mu}\chi_{\nu)} = 0. \tag{10.17}$$

Therefore, we can express Eq. (10.4) as

$$\chi^\nu\nabla^\mu\chi_\nu|_{r=r_+} = \frac{1}{2}\nabla_\mu(\chi^2)|_{r=r+} = -\kappa\chi_\mu|_{r=r_+}, \tag{10.18}$$

where $\chi^2 = \chi^\mu\chi_\mu$. Projecting on some vector ξ which is not normal to χ on the horizon, we obtain

$$\kappa = -\left.\frac{\xi^\mu\nabla_\mu(\chi^2)}{2\xi^\mu\chi_\mu}\right|_{r=r_+}. \tag{10.19}$$

In Eddington-Finkelstein coordinates, we can choose $\xi = \partial_r$ to obtain

$$\kappa = -\left.\frac{\partial_r(\chi^2)}{2\chi_r}\right|_{r=r_+} = \frac{1}{2}\left(\frac{1}{GM} - \frac{GM}{a} + \frac{\sqrt{(GM)^2 - a^2}}{a^2}\right) = \frac{r_+ - r_-}{2(r_+^2 + a^2)}. \tag{10.20}$$

Thus, we obtained the surface gravity of the Kerr black hole. As we learned last week, for static horizons, the surface gravity of a horizon is proportional to its Hawking temperature. Thus, if a QFT is in the inertial vacuum, *i. e.* the vacuum defined by an inertial observer falling into the black hole, an observer hovering above a Kerr black hole is surrounded by a thermal bath of temperature

$$T = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}, \tag{10.21}$$

which in the limit $a \to 0$ recovers the temperature of the Schwarzschild black hole as expected.

(**e**) In the limit $a \to GM$, the surface gravity vanishes. Thus, extremal black holes have temperature $T = 0$. The third law of thermodynamics tells us that it takes an infinite number of steps to lower the temperature to 0. Thus, it is impossible to spin up a black hole to $a = GM$ in a finite number of steps. The third law of black-hole mechanics is indeed that one can't lower the surface gravity of a black hole to 0 by a finite number of steps.

Exercise 11: Ergoregion and Penrose process

Motivation: Rotating black holes might just be the most powerful energy source in the universe, real sci-fi stuff. Here's how.

As you derived in the last exercise, contrary to the Schwarzschild black hole, the timelike Killing vector K is not null on the outer horizon. This does not mean that there are no surfaces, where it satisfies $K^2 = 0$.

(**a**) **Do this exercise before reading on.** Verify that the two vectors $K = \partial_t$ and $R = \partial_\phi$ are Killing vectors of the Kerr metric. Killing vectors are defined such that their covariant derivative is antisymmetric. What physical quantities are conserved due to these symmetries, and how are they related to the black hole parameters M and a?

(**b**) **Do this exercise before reading on:** Find the hypersurface ∂E at which the Killing vector K is null. Verify that it is generically outside the horizon, *i. e.* $r_{\partial E}(\theta) \geq r_+$, where $r_{\partial E}$ parametrizes ∂E.

(**c**) The region bounded by ∂E is called ergoregion. Draw a constant-time hypersurface of the outer horizon and the positive-r branch of the ergosurface ($r_{\partial E}$) as seen from the side, say for $\phi = \pi$, for large a (e. g. $a = 0.9$).

(**d**) The ergoregion is a very special place. There, the norm of the Killing vector K is spacelike. On this basis, argue that timelike observers *cannot* stand still in the ergoregion. This effect is called frame dragging. Which direction do they have to move in? (**Hint:** Note that ∂E is not a horizon. One can leave it.)

(**e**) Consider a particle of mass m which has momentum $p = mu$ with timelike four-velocity u. For simplicity, assume that the particle is only moving on hypersurfaces of constant θ and r, *i. e.* $p = p^t\partial_t + p^\phi\partial_\phi$, and into the future, *i. e.* $p^t > 0$. The particle's energy is defined as $E = K^\mu p_\mu$, its angular momentum as $L = -R^\mu p_\mu$. Are the energy and the angular momentum always positive in the ergoregion outside the horizon, *i. e.* for $r_+ < r_0 < r_{\partial E}$?

(f) Consider that you send an object into the ergoregion, it breaks up into two equal-mass pieces, one of which comes back out. Start out with the original object at infinity such that the conserved total energy equals $E_{\text{tot}} = E_{\text{in}} + E_{\text{out}} = \mu$. Assume the pieces, both of mass m break apart inside the ergoregion, one particle falls into the black hole and one particle escapes to infinity. Use energy conservation to argue that the energy of the outgoing particle can satisfy $E_{\text{out}} > \mu$. Where is the energy extracted from?

(g) Let's maximize the efficiency of this process, *i. e.* the quantity

$$\eta = \frac{E_{\text{out}}}{E_{\text{tot}}}, \tag{11.1}$$

where E_{in} denotes the energy of the infalling particle. Evidently, an efficiency larger than one implies that one gets more energy out of the black hole than went in. Assume that:

- The object breaks apart somewhere outside the outer horizon, *i. e.* at some $r_0 > r_+$.
- The object breaks apart at a turning point in radial motion (*i. e.* $\dot{r} = 0$). At this point during the evolution, on any object of mass m, energy E and angular momentum L, the geodesic equation enforces the constraint

$$\frac{r^2 E}{m} + \frac{2GM}{r}\left(\frac{aE}{m} - \frac{L}{m}\right)^2 + \left(a^2\frac{E^2}{m^2} - \frac{L^2}{m^2}\right) - \Delta = 0. \tag{11.2}$$

- The black hole is extremal, *i. e.* $a = GM$.
- Energy and angular momentum are conserved during the process. This follows from K and R being Killing vectors.

Compute E_{in}, E_{out} and $\eta(m, r_0)$ and maximize η for allowed values of r_0 and m. (**Hint:** An object can be more than the sum of its decay products.)

(h) The Penrose process decreases both the angular momentum J and the mass M of the black hole. Over all, however, J decreases more than M such that $a = J/GM$ decreases. This works until the black hole stops rotating. The mass of a black hole that has been spun down to $a = 0$ by the Penrose process is called the irreducible mass M_{irr}.

Fun fact aka assume without proof: You may have heard of the area theorem stating that the area of black holes cannot decrease in GR. A perfectly administered Penrose process is actually optimal: It does not change the area of the outer horizon.

Show that the area of the Kerr horizon equals

$$A = 4\pi(r_+^2 + a^2). \tag{11.3}$$

Use the fun fact to compute the irreducible mass.

(i) The black hole in the centre of the Galaxy has mass $M \simeq 4 \times 10^6 M_\odot$, where $M_\odot$ denotes the mass of the sun. Imagine that it would be extremal (its actual spin parameter is around $a \in [0.1, .5]$ but never mind reality). Compute how much energy could be extracted from the black hole. You should obtain that it's around 29% of the original black-hole mass. For comparison, nuclear fusion converts 0.7% of matter into energy, and the mass of visible matter in the milky way equals around $\sim 5 \times 10^{10} M_\odot$. Connect the dots. True sci-fi stuff!

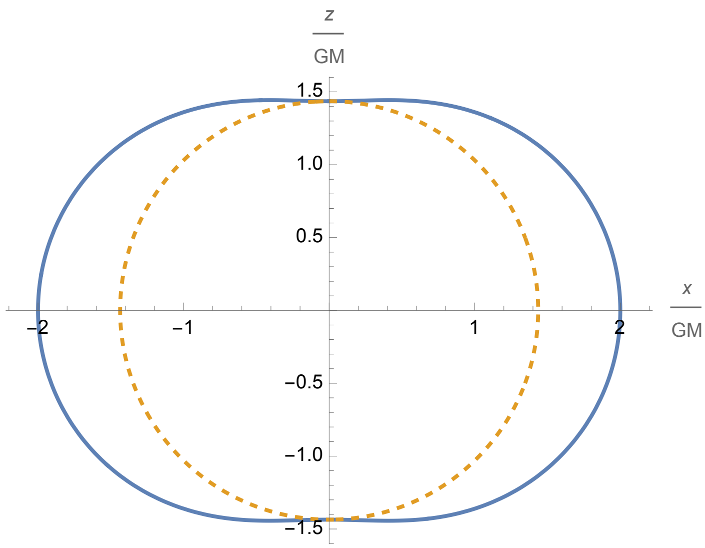

Figure 3: Side view on the ergoregion of a spacelike hypersurface ($t = \text{const}$) of a Kerr black hole with $a = 0.9GM$ given in

(**a**) Killing vector fields satisfy the Killing equation Eq. (13.5) We do this in all generality. Assume that the metric is independent of a coordinate ξ, *i. e.* $\partial_\xi g_{\mu\nu} = 0$. Then, the vector $X = \partial_\xi$ satisfies

$$\nabla_\mu X_\nu = g_{\nu\sigma}\Gamma^\sigma_{\mu\rho}X^\rho = g_{\nu\sigma}\Gamma^\sigma_{\mu\xi} = \frac{1}{2}\left(\partial_\xi g_{\mu\nu} + \partial_\mu g_{\xi\nu} - \partial_\nu g_{\mu\xi}\right). \tag{11.4}$$

However, the metric is independent of ξ, so

$$\nabla_\mu X_\nu = \partial_{[\mu} g_{\nu]\xi} = \nabla_{[\mu} X_{\nu]}. \tag{11.5}$$

Thus, the vector X is a Killing vector. The metric is independent of both t and ϕ. Therefore, K and R are Killing vectors.

(**b**) In Boyer-Lindquist coordinates the norm of the Killing vector K is just $K^\mu K_\mu = g_{tt}$. Thus, K is null when

$$r^2 - 2GMr + a^2\cos^2\theta = 0. \tag{11.6}$$

The most positive root of this equation yields the parametric solution

$$r_{\partial E} = GM + \sqrt{(GM)^2 - a^2\cos^2\theta}, \tag{11.7}$$

which defines the boundary of the ergoregion. Clearly, $r_{\partial E}|_{\theta=0,\pi} = r_+$, but generally $r_{\partial E} \geq r_+$. Thus, the hypersurface where K becomes null lies outside the horizon.

(**c**) I plot the ergoregion for $a = 0.9GM$ in fig. 3.

(**d**) Inside the ergoregion, the Killing vector K becomes spacelike. Since observers must follow timelike worldlines, their 4-velocity cannot be proportional to K. In Boyer–Lindquist coordinates, viz. Eq. (10.1), there is exactly one contribution to the metric which is timelike inside the ergoregion, namely the cross term proportional to $\mathrm{d}t\mathrm{d}\phi$. Thus, the only way to construct a future-directed timelike 4-velocity is to include a nonzero $\mathrm{d}\phi$ component. This means the observer must move azimuthally—i.e., co-rotate with the black hole. This effect is known as frame dragging.

(**e**) Let's compute energy and angular momentum in Boyer-Lindquist coordinates

$$E = g_{tt}p^t + g_{t\phi}p^\phi, \tag{11.8}$$

$$L = -g_{\phi\phi}p^\phi - g_{\phi t}p^t. \tag{11.9}$$

In the physical regime $a < GM$, both K and R are spacelike inside the ergoregion, *i. e.* $g_{tt} < 0$, and $g_{\phi\phi} < 0$. At the same time, the momentum of a timelike observer is timelike, *i. e.* (considering for simplicity an observer with $p^\theta = p^r = 0$)

$$p_\mu p^\mu = g_{tt}(p^t)^2 + 2g_{t\phi}p^t p^\phi + g_{\phi\phi}(p^\phi)^2 = m^2. \tag{11.10}$$

As $g_{tt}, g_{\phi\phi} < 0$ while $g_{t\phi} > 0$, we naturally obtain $p^\phi > 0$, which is just the answer to exercise (**f**).

Given the mass-shell constraint, we can express the energy and the angular momentum as

$$E = \frac{m^2 - g_{\phi\phi}(p^\phi)^2 - g_{t\phi}p^t p^\phi}{p^t} = \frac{m^2 - p^\phi(|g_{t\phi}|p^t - |g_{\phi\phi}|p^\phi)}{p^t}, \tag{11.11}$$

$$L = -\frac{m^2 - g_{tt}(p^t)^2 - g_{t\phi}p^t p^\phi}{p^\phi} = -\frac{m^2 - p^t(|g_{t\phi}|p^t - |g_{tt}|p^\phi)}{p^t}. \tag{11.12}$$

Thus, if p^ϕ and m^2 are chosen sufficiently small while at the same time satisfying Eq. (11.10), the energy is negative, and so can be the angular momentum. A concrete example is $a = GM/2$, $r = 1.9GM$, $\theta = \pi/2$, $p^t = 34.3m$ and $p^\phi = .24/G$ yielding $E = -0.05M$ and $L = -0.82GM^2$.

(**f**) If the piece falling into the black hole is allowed to acquire negative energies, *i. e.* $E_{\text{in}} < 0$, the outgoing piece can have more energy than the whole object had before falling in because

$$E_{\text{out}} = E_{\text{tot}} - E_{\text{in}} = \mu + |E_{\text{in}}| > \mu. \tag{11.13}$$

Thus, energy has been extracted from somewhere. The only place this energy can come from is the black hole itself, which will lose rotational energy and mass by this process to match the negative energy and the negative angular momentum of the infalling particle. Note that due to the area theorem, namely that black hole horizon areas don't decrease, this loss has to be such that the change in $a = J/M$ is negative.

(**g**) First, we solve the constraint from the geodesic equation at r_0 for the total angular momentum obtaining

$$L_{\text{tot}} = \frac{2aGM \pm \sqrt{2GMr_0\Delta}}{2GM - r_0}\mu. \tag{11.14}$$

As this is also the place where the object breaks into two pieces, the geodesic equation equivalently applies to those two. Thus, for both particles we obtain the angular momenta

$$L_{\text{in/out}} = \frac{2aGME_{\text{in/out}} \pm \sqrt{r_0\Delta[E_{\text{in/out}}^2 r_0^2 + m^2(2GM - r_0)]}}{2GM - r_0}. \tag{11.15}$$

At the same time, we know that $E_{\text{tot}} = \mu$. Thus, we can use energy conservation and angular-momentum conservation to determine the individual energies $E_{\text{in/out}}$. Thus, imposing

$$E_{\text{tot}} = E_{\text{in}} + E_{\text{out}}, \qquad L_{\text{tot}} = L_{\text{in}} + L_{\text{out}}, \tag{11.16}$$

we obtain (choosing the roots $E_{\text{in}} < E_{\text{out}}$)

$$E_{\text{in/out}} = \frac{\mu}{2}\left[1 \mp \sqrt{\left(1 - \frac{4m^2}{\mu^2}\right)\frac{2GM}{r_0}}\right]. \tag{11.17}$$

Thus, we obtain the efficiency

$$\eta = \frac{1}{2}\left[1 + \sqrt{\left(1 - \frac{4m^2}{\mu^2}\right)\frac{2GM}{r_0}}\right]. \tag{11.18}$$

The efficiency is maximal for minimal r_0, *i. e.* when hovering just above the horizon, and minimal m, *i. e.* the object decays into massless particles like photons. For $a = GM$, $m = 0$, and $r_0 = r_+ = GM$, we obtain

$$\eta = \frac{1}{2}\left[1+\sqrt{2}\right] \simeq 120\%. \tag{11.19}$$

Thus, we get out 120% of the energy we sent into the black hole.

(**h**) First we need to compute the area of the horizon. The horizon is a surface of constant time ($\mathrm{d}t = 0$) and radius $r = r_+$ (thus, also $\mathrm{d}r = 0$). Thus, the induced metric on the horizon reads

$$\mathrm{d}s^2_{(2)} = -\rho^2(r_+)\mathrm{d}\theta^2 - \left[(r_+^2 + a^2)^2 - a^2\Delta\sin^2\theta\right]\frac{\sin^2\theta}{\rho^2(r_+)}\mathrm{d}\phi^2. \tag{11.20}$$

The area is then defined as

$$A \equiv \int_{\mathrm{H}} \sqrt{h}\mathrm{d}^2x, \tag{11.21}$$

where h is the determinant of the induced metric on the horizon surface. Thus, we have to compute

$$A = (a^2 + r_+^2)\int_0^{\pi}\int_0^{2\pi}\sin\theta\mathrm{d}\phi\mathrm{d}\theta = 4\pi(a^2 + r_+^2). \tag{11.22}$$

The irreducible mass amounts to the black hole not rotating any more such that

$$A = 16\pi(GM_{\mathrm{irr}})^2. \tag{11.23}$$

As the black-hole area does not change, this equals the area before all of its rotational energy has been extracted. Thus, we obtain

$$M_{\mathrm{irr}} = \frac{1}{2G}\sqrt{a^2 + r_+^2} = \sqrt{\frac{M^2 + M\sqrt{M - a/G}}{2}}. \tag{11.24}$$

In particular, for extremal black holes the irreducible mass becomes minimal, yielding

$$M_{\mathrm{irr}} = \frac{M}{\sqrt{2}}. \tag{11.25}$$

(**i**) The amount of energy that could be extracted is

$$M - M_{\mathrm{irr}} = \left(1 - \frac{1}{\sqrt{2}}\right)M \simeq 0.29M. \tag{11.26}$$

This amounts to 29% of the original mass, which is an extremely large amount.

Throughout their lifetime, all stars in the milky way generate 0.7% of their mass in radiation from nuclear fusion. The mass of all stars in the galaxy being around $5 \times 10^{10} M_\odot$, all in all this amounts to a radiated energy of $E_{\mathrm{stars}} \sim 4 \times 10^8 M_\odot$. If the black hole at the centre of the Milky Way was extremal, we could extract $E_{\mathrm{Penrose}} \sim 10^6 M_\odot$, which is just about 400 times smaller. Besides, that black hole is rather small as supermassive black holes go. This is probably the strongest energy source in the universe we know of.