

Lecture notes on Quantum Field Theory

Astrid Eichhorn,
notes typeset by F. Homburg*

Wintersemester 2025/26

Contents

0 Preliminaries	2
1 Introduction	3
1.1 Motivation: Why quantum field theory?	3
1.2 Why learning quantum field theory is hard	4
1.3 Why learning quantum field theory is absolutely worth it	4
1.4 Classical field theory	5
2 The importance and the mathematics of symmetries	9
2.1 Symmetries are described by groups	9
2.2 Lie groups	11
2.3 Lorentz transformations and the Lorentz group	15
2.4 Poincaré group and why we classify particles by their mass and spin	18
2.5 Noether's theorem	21
3 Canonical quantization of the free scalar field	28
3.1 Canonical Quantization of the real scalar field in the Schrödinger picture	28
3.2 Creation and annihilation operators and construction of the Fock space	30
3.3 Casimir effect	34
3.4 Quantum statistics	36
3.5 Interpretation of $\phi(\vec{x})$	37
3.6 Heisenberg picture	37
3.7 Causality	38
3.8 A note on quantum entanglement:	39
3.9 Propagators and causality	39
4 Path-integral quantization for the scalar field	43
4.1 Path-integral for Quantum Mechanics	44
4.2 Classical action, least-action principle	49
4.3 Time-ordered products and generating functional	49
4.4 Functional differentiation	51
4.5 Projection onto the ground state at asymptotic times	52
4.6 Path integral in quantum field theory	54
5 The quantum effective action $\Gamma[\Phi]$ and interacting quantum field theory	55
5.1 Effective potential in scalar $\lambda\phi^4$ theory	57

*This is a draft version of the lecture notes. If you notice typos, points that remain unclear, mistakes or similar, please send an email to f.homburg@thphys.uni-heidelberg.de and a.eichhorn@thphys.uni-heidelberg.de.

0 Preliminaries

Literature

These lecture notes combine material from several sources. Some topics in these lecture notes are treated in all books listed below; for some, a particular book or selection of books is better suited. In this case, this will be indicated in the corresponding chapter.

In particular, you might find it useful to also read up on the topics in the following books as well as lecture notes (available online):

- Srednicki, Quantum Field Theory
- Ryder, Quantum Field Theory
- Gelis, Quantum Field Theory
- Schwartz, Quantum Field Theory and the Standard Model
- Peskin/Schröder, Quantum Field Theory
- Nastase, Quantum Field Theory
- Fradkin, Quantum Field Theory
- There are many other books on QFT and it is often a matter of personal taste, which one is most useful.
- Lecture notes on QFT by D. Tong (Cambridge University), A. Hebecker (Heidelberg University), T. Weigand (from the QFT courses at Heidelberg University)

Many topics are treated to the greatest level of depth in the QFT books by Weinberg. However, for a first encounter with a topic, the books are usually not useful, but rather become helpful later on, when one has already learned about a topic and wants to come back to it to learn more about it.

There is also the book “Quantum field theory in a nutshell” by Zee, which focuses more on some conceptual aspects rather than technical points and it can be a useful addition to the above list of literature.

Mini-exercises

The best way to learn quantum field theory is to do calculations yourself, and think and discuss about concepts yourself. Therefore, each lecture has at least one “mini-exercise”, which you will work on during the lecture. This gives you the opportunity to engage more actively with the material and notice when you have questions. You will likely not always have time to finish the mini-exercise during class. Therefore, solutions will not only be provided on the blackboard, but are also available in the back of the lecture notes. They will be made available in the update of the lectures notes that will be made online after each lecture.

1 Introduction

1.1 Motivation: Why quantum field theory?

Quantum mechanics is a non-relativistic theory. This results in a question, namely:

- What happens to systems in which quantum effects and relativistic effects are important?
There is a heuristic argument that points us towards how relativistic quantum physics differs from quantum mechanics. From the standard Heisenberg uncertainty principle, one can motivate an uncertainty relation between energy E and time t , namely $\Delta E \Delta t \geq \frac{\hbar}{2}$. In a relativistic setting, we can combine this with $E = mc^2$, which we know from special relativity.
⇒ We expect that particle number is never fixed in a system, because, for short enough time durations, energy is not constant, but fluctuates and these fluctuations in energy translate into fluctuations in particle number. We call these fluctuations “virtual” particles.
⇒ We cannot work with a wavefunction for a fixed number of particles, as we did in quantum mechanics. Instead, we need a formalism in which the particle number can change in a system over time, and in which the presence of virtual particles is accounted for.

We can also see the incompatibility between special relativity and quantum mechanics in a different way:

- Special relativity requires that two measurements that are done at spacelike separation, must be independent in order not to violate causality. In Quantum Mechanics, independence of measurements is encoded in commuting operators. However, the notion that spacelike separated operators commute is not naturally built into QM.
⇒ We need to adapt our formalism.

How should the new formalism look like?

To go beyond wavefunctions for fixed numbers of particles, we need a (mathematical) quantity that is more fundamental than particles, i.e., particles should be a *derived* notion.

We take inspiration from electrodynamics, because electrodynamics can be formulated in a relativistic way. At the same time, we know from the photo-electric effect, that there are particles in electrodynamics, namely photons. Thus, it is a useful guide to point us to the type of formalism that we should develop. Electrodynamics is a field theory, i.e., the fundamental quantity is a *field*, i.e., a quantity that takes on values at each spacetime point.

From experiments, we already know that photons (the corresponding particles) are derived from the field, in fact, they correspond to (quantized) excitations of the field. This can, e.g., be seen in laser experiments, in which the power incident on a screen is recorded. As the intensity of the laser is lowered, the power arrives in discrete, “quantized packages”, the photons.

In order to be compatible with special relativity, we need to build a theory which has Lorentz invariance built into it, just like the relativistic formulation of electrodynamics has.

What will this type of theory be able to describe?

- elementary particles and their interactions, in particular the Standard Model of particle physics.

- any setting in which particle number is not conserved, e.g., condensed-matter-systems in which we are interested in effective (not fundamental) excitations, such as, e.g., phonons, or Cooper-pairs in superconductivity.
- if the energy of the system is low enough, the formalism that we are developing is even sufficient to understand the quantum properties of gravity.

Note: in our current understanding of cosmology, the origin of all structures in the universe (galaxies, galaxy clusters ...) are quantum fluctuations of the fundamental fields in the early universe. Ultimately, we thus owe our existence to the physics of QFT!

1.2 Why learning quantum field theory is hard

Quantum field theory is not an easy subject. This has several reasons. First, the quantities that we are dealing with are often abstract and more difficult to develop an intuitive understanding of than, for instance, systems in classical mechanics. Second, we need to develop an entirely new formalism to describe quantum fields, in which we bring together classical field theory and quantum theory. In other words, we are learning a (mathematical) language in which to describe the systems that we are interested in, and, just like with any other new language, learning it can be hard and it takes some time until the concepts start to feel familiar and intuitive.

However, you should not feel discouraged by this or think about giving up. Rather, if you have questions and/or doubts, bring them up with the lecturer (either after the lecture, or by email to eichhorn@thphys.uni-heidelberg.de) or to your tutor, or to the head tutor, Zois Gyftopolous (gyftopolous@thphys.uni-heidelberg.de). The whole team of lecturer and tutors is here to support you in learning and understanding quantum field theory!

1.3 Why learning quantum field theory is absolutely worth it

Quantum field theory provides the framework for the most advanced and deepest understanding of fundamental physics that we have. Therefore, it is like a key with which we can unlock fascinating insights into elementary particles and their properties. Thus, some of the highlights that await us this term are:

- understanding how powerful symmetries are and how we can deduce properties of elementary particles from an understanding of the Lorentz group and how we can deduce the existence of the electromagnetic field from thinking about symmetries
- understanding where the Pauli principle for Spin-1/2-particles comes from
- understanding why antiparticles must exist in order for causality to not be violated
- understanding that the vacuum is not a boring state of “nothing”, but is a highly non-trivial state which results in a force between conducting plates (“Casimir force”) or the scattering of photons off each other (unlike in classical electrodynamics, in which the equations of motion for the gauge field are linear and electromagnetic waves do not interact)
- and much more!

1.4 Classical field theory

We have already emphasized the role and importance of *symmetries*, so we will spend some more time developing the *mathematics of symmetries*, namely groups and their representations. First, however, we need to establish some of the notions that form the basis of this course, namely fields and their classical description.

A field takes a value at each spacetime point. Examples that you may already know include

- the E- and B-field, $E(\vec{x}, t), B(\vec{x}, t)$, which are 3-vectors.
- the density in hydrodynamics, $\rho(\vec{x}, t)$, which is just a one-component function.
- the gauge field $A_\mu(\vec{x}, t)$ in electrodynamics, which is a 4-vector.

To describe their dynamics, we start from an action S , which is a *functional*, i.e., its argument is a function (and it maps to the real numbers).

For instance, in the relativistic way of writing electrodynamics, we have

$$\begin{aligned} S[A] &= \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}, \\ &= \int d^4x \mathcal{L}_{\text{ED}}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{1}$$

We denote functionals with square brackets around their arguments, which are functions. $\mathcal{L}_{\text{ED}}$ is the *Lagrange density*. It is not a functional, because it does not depend on the full function (in this case, the field at all spacetime points), but is just a function of the spacetime-coordinates, through its dependence on the field at a point.

To establish some of the key notions, we will use a scalar field, conventionally denoted by $\phi(\vec{x}, t)$. An example for scalar fields relevant in nature is the Higgs field in the Standard Model; hypothetical scalar fields include the inflaton field (that drives the (conjectured) inflationary phase in the early universe), and proposals for dark matter (e.g., the axion, which is, to be more precise, a pseudoscalar). Scalars that can be collective degrees of freedom also play a role in many condensed-matter systems, starting from the Ising model.

The Lagrange density $\mathcal{L}$ depends on the field $\phi(\vec{x}, t)$ and its derivatives, $\partial_\mu \phi(\vec{x}, t)$, $\partial_\mu \partial_\nu \phi(\vec{x}, t)$... and is a priori completely arbitrary. We will make two assumptions:

- the Lagrange density is *local*, i.e., it depends on fields and their derivatives at one point and it only depends on a finite number of derivatives. (We call this local, because a derivative always compares a field at a point to its (infinitesimally removed) neighboring point. An infinitely high power of derivatives thus involves fields a finite distance apart.)

This has two motivations: First, observationally, local interactions seem to describe nature very well; e.g., in the LHC detectors, one can see that particles interact locally. Second, non-local interaction may get into conflict with causality, because non-localities may mean interactions at spacelike distances.

- We assume that the Lagrange density does not have higher than second derivatives in time. The reason is Ostrogradsky's theorem, which is a theorem in classical mechanics and states that, (under a non-degeneracy condition), a Hamiltonian that contains higher-than-second-order time derivatives is unbounded from below. This may- but need not!- make the the-

ory dynamically unstable. Because this theorem implicitly underlies the formulation of Lagrangians in many settings (classical mechanics, classical field theory, quantum field theory), we will take a closer look at it in the exercises.¹

The Lagrangian

$$L = \int d^3x \mathcal{L}, \quad (2)$$

is the spatial integral of the Lagrange density. We will often work with $\mathcal{L}$, because it makes the equal treatment of space and time, that we want in a relativistic theory, manifest. It is often called “the Lagrangian” in a slight abuse of naming conventions.

$\mathcal{L}$ consists of two parts, a kinetic part, T , that depends on derivatives, and a potential, V ,

$$\mathcal{L} = T - V. \quad (3)$$

We will often focus on

$$T = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi, \quad (4)$$

and

$$V = \frac{1}{2} m^2 \phi^2 + \lambda \phi^4, \quad (5)$$

where in V we assumed that we can Taylor-expand $V(\phi)$ around a minimum ϕ_0 and we can set $\phi_0 = 0$ and $V(\phi_0) = 0$ without loss of generality. We further assume a symmetry $\phi \rightarrow -\phi$, so that there is no ϕ^3 present, which would render $V(\phi)$ unbounded from below. We call m the mass, because we will see that the equations of motion imply $p^2 = m^2$ for the square of the four-momentum, if the term $m^2 \phi^2$ is present in the Lagrangian. The quartic term, $\lambda \phi^4$ leads to non-linear equations of motion, i.e., it describes *interactions* of the field (and the corresponding particles) with itself. The strength of these interactions is parameterized by the coupling λ . In the next few lectures, we focus on just the mass term.

Our choice of T requires a bit more justification: The kinetic part describes how the field changes in space and time, thus it must contain a derivative, and $\partial_\mu \phi$ is the building block to use. In order to have a Lorentz-invariant expression, we must contract the open index and the only other 4-vector we have is another derivative. Thus, up to rescalings of the term, we have a unique lowest order action in ϕ

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial_\nu \phi \eta^{\mu\nu} - \frac{m^2}{2} \phi^2 \right), \quad (6)$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ in our conventions, which most QFT books use. Many GR books use $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The overall sign is pure convention; the difference in signs between the time part and the spatial part is physics.

Mini-Exercise 1. We made the statement that we can set a constant and a linear term in $\mathcal{L}$ to zero without loss of generality. For the constant term, this is because the equations of motion follow from minimizing the action and the field value that minimizes S does not depend on whether or not there is a constant shift in S .

¹You have probably encountered or will encounter many examples where the Lagrangian does not have higher than second order time derivatives. Electrodynamics is one example, General Relativity another, and classical mechanics is full of examples. Note however that there are subtleties and there are counterexamples to the intuition that a Hamiltonian that is unbounded from below leads to instabilities.

For the linear term, we can always remove it by a change of our field variable (which you can think of as analogous to a change in coordinates in class. mech.)

Show this! Start with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - C\phi - \frac{1}{2}m^2\phi^2. \quad (7)$$

Define $\varphi = \phi + \gamma$. What is the choice of γ , such that

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \text{const} ? \quad (8)$$

Solution.

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - C(\varphi - \gamma) - \frac{1}{2}m^2(\varphi - \gamma)^2 \\ &= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \underline{C\varphi} + C\gamma - \frac{1}{2}m^2\varphi^2 + \underline{m^2\varphi\gamma} - \frac{1}{2}m^2\gamma^2 \end{aligned}$$

Define $\gamma = \frac{C}{m^2}$:

$$\rightarrow \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{C^2}{m^2} - \frac{1}{2}m^2\varphi^2 - \frac{1}{2}m^2\frac{C^2}{m^2}.$$

The equations of motion follow from extremizing the action, i.e., we perform a variation of the action (i.e., a variation of the field, $\phi \rightarrow \phi + \delta\phi$, by some arbitrary amount $\delta\phi$). We set the variation of the action to zero, just like, when we are searching for the minimum of a function, we are setting its first derivative (analogous to the variation of the argument of the function) to zero:

$$\begin{aligned} 0 = \delta S &= \delta \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \right) \\ &= \int d^4x \left((\partial_\mu\phi)\eta^{\mu\nu}(\partial_\nu\delta\phi) - m^2\phi\delta\phi \right) \\ &= \int d^4x \left(-(\partial_\nu\partial_\mu\phi)\eta^{\mu\nu}\delta\phi - m^2\phi\delta\phi \right) \\ &= \int d^4x \left(-(\partial_\nu\partial_\mu\phi)\eta^{\mu\nu} - m^2\phi \right) \delta\phi, \end{aligned} \quad (9)$$

where in the second-to-last step we used partial integration and where we assume that $\delta\phi = 0$ at $x \rightarrow \pm\infty$. Because $\delta\phi$ is an arbitrary variation, to satisfy Eq. (9), the factor $-\partial_\nu\partial_\mu\phi\eta^{\mu\nu} - m^2\phi$ must be zero.

This is the Klein-Gordon equation,

$$\boxed{\partial^2\phi + m^2\phi = 0}, \quad (10)$$

with $\partial^2 = \partial_\mu\partial^\mu$. The Klein-Gordon equation is a relativistic, massive wave equation.

For the Lagrangian, $\delta S = 0$ translates into the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \right) = 0. \quad (11)$$

The solutions to the equations of motion are spanned by plane waves,

$$\phi(x) = \phi_0 \cos(kx), \quad (\text{assuming } \phi(x) = \phi(-x)) \quad (12)$$

with the shorthand $kx = k_\mu x^\mu$ and the relativistic, massive dispersion-relation $k_\mu k^\mu = k^2 = m^2$.

Later on, a starting point for one quantization scheme (path-integral quantization) will be the action, but the starting point for another quantization scheme (canonical quantization) will be the Hamiltonian.

Just as in classical mechanics, where we define $p = \frac{\partial L}{\partial \dot{q}}$, the canonically conjugate momentum, and $H(p, q) = p\dot{q} - L$, in quantum field theory we define $\pi(\vec{x})$, the canonically conjugate field. (Note: it is the canonically conjugate field to ϕ , but has nothing to do with the momentum of the particles that we will describe. It is sometimes called the (canonically conjugate) momentum field, because it arises in the generalization of the Hamiltonian formalism to QFT and it generalizes the momentum of a particle, which is the canonically conjugate variable to the position.) Its definition is

$$\pi(\vec{x}) = \frac{\delta L}{\delta \dot{\phi}(\vec{x})}, \quad (13)$$

which is a functional derivative, i.e., a derivative with respect to a function. Just like $\frac{\partial x}{\partial x} = 1$, we have

$$\frac{\delta \phi(\vec{x})}{\delta \phi(\vec{y})} = \delta^3(\vec{x} - \vec{y}). \quad (14)$$

Thus, for the Lagrangian in Eq. (6), we obtain

$$\begin{aligned} \pi(\vec{x}) &= \frac{\delta}{\delta \dot{\phi}(\vec{x})} \int d^3y \left(\underbrace{\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\vec{\nabla} \phi)^2}_{\partial_\mu \phi \partial^\mu \phi} - \frac{1}{2} m^2 \phi^2 \right) \\ &= \int d^3y \left(\dot{\phi} \delta^3(\vec{x} - \vec{y}) \right) = \dot{\phi}(\vec{x}). \end{aligned} \quad (15)$$

Thus, to calculate the Hamiltonian, we can use that $\dot{\phi}$ can be substituted by π . We obtain the Hamiltonian of the system as

$$\begin{aligned} H &= \left(\int d^3x \pi \dot{\phi} \right) - L \\ &= \int d^3x \left(\pi^2 - \left(\frac{\pi^2}{2} - \frac{1}{2} (\vec{\nabla} \phi)^2 - \frac{m^2}{2} \phi^2 \right) \right) \\ &= \frac{1}{2} \int d^3x \left(\underbrace{\pi^2 + (\vec{\nabla} \phi)^2}_{2\mathcal{H}} + m^2 \phi^2 \right) \\ &= \int d^3x \mathcal{H}, \end{aligned} \quad (16)$$

where we defined the Hamiltonian density $\mathcal{H}$.

2 The importance and the mathematics of symmetries

Useful literature for this chapter is the following: There are books on group theory in physics and more specifically particle physics, e.g., “Group theory in physics” by Wu-Ki Tung and “Lie algebras in particle physics” by Howard Georgi.

QFT books also cover discussions of symmetry groups, for instance: Schwartz, chapter 2, covers the basics of Lorentz transformations, and group theory basics for the Lorentz group are discussed in 10.1. The Lorentz group and its Lie algebra generators are also discussed in Srednicki, chapter 2. Gelis (chapter 7.1) summarizes Lie groups and Lie algebras.

Symmetries are one of the most important foundational elements in QFT. This becomes obvious from many examples:

- i) In particle physics, the various mesons and baryons are organized into sets, e.g., the eight lightest mesons are grouped into the meson octet according to the “eightfold way” which is based on a so-called “ $SU(3)$ flavor symmetry”. Historically, this type of organization into sets according to symmetries was central in predicting new particles.
- ii) You might have heard that the Standard Model is an $SU(3) \times SU(2) \times U(1)$ gauge theory. Specifying this symmetry already fixes a large part of the Standard Model particle content and the allowed interactions between particles.
- iii) In condensed matter, phase transitions are associated with spontaneous *breaking of symmetries*. For instance, in a ferromagnet, at high enough temperature, there is no macroscopic magnetization, which means that there is full rotational symmetry for each of the microscopic spin vectors. At low temperature, in the magnetized phase, rotational symmetry is broken, because the macroscopic magnetization spontaneously selects one spatial direction. More generally, by knowing the symmetries that the degrees of freedom in a condensed-matter system obey, we can already figure out which phases and phase transitions there could be.
- iv) Lorentz symmetry (or its generalization, Poincaré symmetry, which adds translations (in space and in time)) determine much of the properties of elementary particles and their interactions and much of the mathematical structure of QFT.
For instance, the fact that we characterize elementary particles by their mass and their spin is a direct consequence (as we will work out) follows from considering the Poincaré group.
- v) ...

This motivates us to dive into the *mathematics of symmetries*, because this appears to be the language in which large parts of nature can be described.

2.1 Symmetries are described by groups

It turns out that there exists a mathematical structure that is exactly adapted to formalizing symmetries, and that is a group.

Definition of a group:

A group G is a set of elements $G_i \in G$, together with a “multiplication” $\cdot$, such that

$$G_i \cdot G_j = G_k, \quad G_k \in G \quad \forall G_i, G_j \in G. \quad (17)$$

This means that we can combine two elements of the group by the multiplication and we obtain another element of the group. The multiplication law satisfies

- associativity

$$(G_i \cdot G_j) \cdot G_k = G_i \cdot (G_j \cdot G_k) \quad \forall G_i, G_j, G_k \in G. \quad (18)$$

- $\exists$ identity element E , s.t.

$$G_i \cdot E \in G \quad \forall G_i \text{ and } E \in G. \quad (19)$$

- inverse element

$$\forall G_i \in G \exists G_i^{-1} \in G, \text{ s.t. } G_i \cdot G_i^{-1} = E. \quad (20)$$

Note that the identity element is unique, as is the inverse for each element.

Let's parse this definition and the intuition behind the various requirements in physics language, using rotations as an example and thinking of a spherically symmetric system:

- Two rotations can be performed consecutively, yielding a third rotation (about a different axis). This is the multiplication law which allows us to combine group elements into new group elements.
- When three rotations are performed, either the 1st and 2nd or 2nd and 3rd can be combined, such that the consecutive execution of the three of them is equal in any of the two combinations. (Note that we must not reverse the *order* of the three rotations, because the group is not commutative.)
- There is an identity element, namely rotation by 0° (or no rotation).
- For each rotation, we can reverse the sense of rotation to rotate back, such that the combination of rotation and inverse rotation yields no rotation.

You may already know that rotations can be represented by matrices, such that, e.g., the identity is the unit matrix and the inverse element is the inverse matrix.

We will encounter two *mathematically distinct* sets of groups that encode symmetries in QFT:

- 1) discrete groups (with a finite set of elements), for instance reflections about a plane (has three elements: the reflection, its inverse, and the identity).
- 2) continuous groups, which are Lie groups. The rotation group is an example. It is continuous, because it has infinitely many group elements (rotations by different angles) and “neighboring” rotations only differ infinitesimally.

We will also encounter three *physically distinct* types of groups²

²There is a theorem, the Coleman-Mandula theorem, that says that, under some assumptions, there are no symmetry groups that mix spacetime symmetry transformations with internal symmetry transformations. The realization that, by violating the assumptions, one can get around this theorem, and is then required to introduce so-called “super-partners” led to the development of supersymmetry, which we will not treat in this course, but which is a very interesting mathematical developments worth understanding. In nature, supersymmetry is realized in some low-dimensional settings in condensed-matter theory, but does not appear to be realized in particle physics. It is, however, instrumental in one approach to quantum gravity, namely in string theory.

- a) spacetime symmetry groups, which can either be continuous (like the Lorentz group, $SO(1,3)$), or discrete (like time-reversal symmetry which maps the time t to $-t$)
- b) internal symmetry groups, where “internal” here means that the symmetry does not act on space and time (like, e.g. the Lorentz group), but only on the field. These come in two different versions:
 - i) global internal symmetries (like the $\mathbb{Z}_2$ -symmetry $\phi \rightarrow -\phi$ that we imposed on scalar field theory to ensure that there is no ϕ^3 term in the scalar potential, or the $SO(3)$ symmetry that is imposed on the scalar field in the Heisenberg model that describes phase transitions in certain materials).
Global means that the symmetry transformation is the same for the field at all spacetime points.
 - ii) local internal symmetries (like the $U(1)$ gauge symmetry of electromagnetism).
Local here means that the symmetry transformation can be different at different spacetime points (even if it doesn’t act on the spacetime itself).

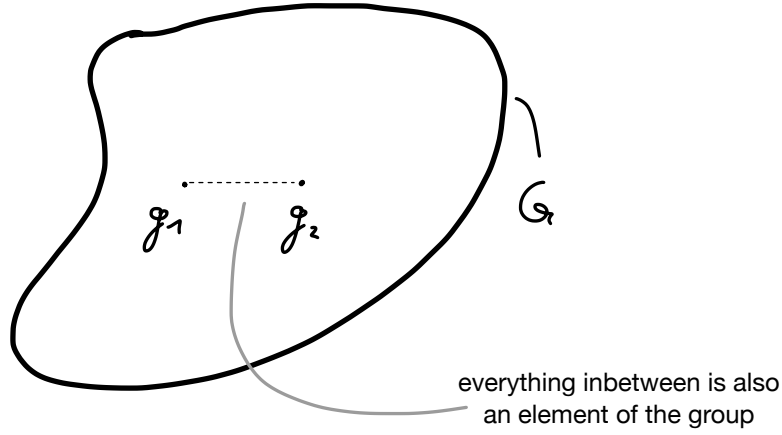
Some of these notions may seem a little abstract at the moment. They will become clearer as we develop our understanding of group theory and come up with examples.

The most relevant groups for us will be *Lie Groups*.

2.2 Lie groups

These are groups in which the group elements form not just a set, but a differentiable manifold (which is a collection of points such that each point has an open neighborhood that is equivalent to $\mathbb{R}^n$ and which can be covered by coordinate charts that overlap partially).

This means that the group is *continuous*, such that you can always find a group element infinitesimally close to any given element. Intuitively, we can see directly that the group of rotations should be such a continuous group, because we can always rotate by an arbitrarily small amount and thus find rotations which are only infinitesimally different from each other.



Examples:

- $U(1)$ is the group of all unitary 1×1 matrices, i.e.,

$$G = e^{i\alpha}, \quad \alpha \in \mathbb{C}. \quad (21)$$

The corresponding manifold is the circle (of radius 1) in the complex plane.

As a global symmetry, the phase α of the transformation does not depend on the spacetime point. As a local symmetry, α is upgraded to a function $\alpha(x^\mu)$. We will explore the consequences of this soon. In fact, this group determines the properties of photons and their interactions with charged particles.

- $SU(2)$ is the group of 2×2 unitary matrices with determinant 1. The corresponding manifold is the 3-sphere, S^3 .

To see this, we write

$$U^\dagger U = \mathbb{1} \implies U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with } |a|^2 + |b|^2 = 1 \text{ for } a, b \in \mathbb{C} \quad (22)$$

(Check:

$$U^\dagger = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \quad \text{and} \quad U^\dagger U = \begin{pmatrix} |a|^2 + |b|^2 & \cancel{a^* b} - \cancel{b a^*} \\ \cancel{b^* a} - \cancel{a b^*} & |a|^2 + |b|^2 \end{pmatrix} = \mathbb{1}) \quad (23)$$

Now we write both complex numbers in terms of real and imaginary part,

$$a = x + iy, \quad b = z + it \quad (24)$$

$\implies |a|^2 + |b|^2 = |x|^2 + |y|^2 + |z|^2 + |t|^2 = 1$ parametrizes the group manifold $SU(2)$, where $x, y, z, t \in \mathbb{R}$.

This is the equation defining a unit 3-dimensional sphere embedded in 4-dimensional space, i.e., S^3 .

$SU(2)$ is the symmetry group determining the properties of the weak gauge bosons (W^+ , W^- and Z) and their interactions with the fermions in the Standard Model.

With a Lie group comes a Lie algebra $\mathfrak{g}$, $\text{Lie}(G) = \mathfrak{g}$. Knowing about the Lie algebra is useful, because all properties of the Lie group follow from knowing the so-called *generators* of the Lie algebra and their commutation relations.

A *Lie algebra* is a vector space $\mathfrak{g}$ with a bilinear, antisymmetric map:

$$\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}, \quad (a, b) \mapsto [a, b] = -[b, a] \quad (25)$$

(that we suggestively write in the same notation that we use for the commutator) that satisfies the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0. \quad (26)$$

We will only need matrix groups and matrix algebras. For matrix Lie groups, the relation between group and algebra is given through the *exponential map*: For $a_i \in \mathfrak{g}$,

$$G_i = \exp(a_i), \quad (27)$$

(defined through its Taylor series) is a group element. Each group element (in the so-called identity component of G) can be written in such a way. For $0 \in \mathfrak{g}$, $\mathbb{1} = \exp(0) \in G$.

We can find a basis in $\mathfrak{g}$ and these elements of the Lie algebra are called the *generators*. Having this basis of generators, we can construct every group element through the exponential map.

Example: Rotation group $SO(3)$

$SO(3)$ is the group of special orthogonal 3×3 matrices, i.e., matrices which are orthogonal, so $\text{Rot} \text{Rot}^\top = \mathbb{1}$, where $\text{Rot}^\top$ denotes the transposed matrix, and special, i.e., their determinant is $+1$. They describe rotations, because we can check that the requirement that a rotation leaves the length of a vector invariant requires $\text{Rot} \text{Rot}^\top = \mathbb{1}$. To check this, consider a spatial vector, with components x^i . Under a rotation, it is mapped to

$$x^i \rightarrow x^{i'} = \text{Rot}^i_k x^k. \quad (28)$$

We require that its length stays invariant, so that

$$x^i x^j \delta_{ij} = x^{i'} x^{j'} \delta_{ij} = \text{Rot}^i_k \text{Rot}^j_l x^k x^l \delta_{ij}. \quad (29)$$

Thus, $\mathbb{1} = \text{Rot}^i_k \text{Rot}^j_l \delta_{ij} = \text{Rot}^T \text{Rot}$. This is in particular realized by matrices of the form

$$\text{Rot}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}, \quad (30)$$

and analogously for rotations about the y and the z -axis.

Claim: $\text{Lie}(G) = \{\text{antisymmetric } 3 \times 3 \text{ matrices}\}$

Mini-Exercise 2. Check that $\text{Rot} \text{Rot}^\top = \mathbb{1}$, as required for $\text{Rot} \in SO(3)$, is realized by $\text{Rot} = \exp(T)$, if $T_{ij} = -T_{ji}$.

Solution.

$$\text{R} \text{R}^\top = \exp(T) \exp(-T) = \mathbb{1},$$

because

$$\text{R}^\top = (\exp(T))^\top = \exp T^\top = \exp(-T).$$

Let's see how we can reconstruct the group elements, i.e., the Rot matrices, from the Lie algebra generators. An antisymmetric 3×3 matrix with real components (so that Rot is real), has 3 independent components, so we have three basis elements

$$T_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (31)$$

and analogously for the other two *generators* of the Lie algebra. Now we can write a rotation about the x -axis as

$$\text{Rot}_x = \exp(\theta T_x) = \mathbb{1} + \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \mathcal{O}(\theta^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta \\ 0 & \theta & 1 \end{pmatrix} + \mathcal{O}(\theta^2), \quad (32)$$

which is clearly the infinitesimal version of the rotation matrix given above.

What will be crucial in our construction of QFT is the notion of *representations* of groups and algebras. For instance, we will construct the spin-0, spin-1/2 and spin-1 representations of the Lorentz group to describe the Higgs field, the electron and the photon in the Standard Model, or various excitations in condensed-matter systems.

Intuitively, a representation is a set of objects which satisfy the same multiplication rules as the abstract group elements, i.e., they are often matrices, for which the multiplication satisfies the combination rules that the group elements satisfy.

More formally, a representation R of a group is a map $G \xrightarrow{R} \text{GL}(V)$ (where $\text{GL}(V)$ are the general linear transformations on a vector space), such that $R(\mathbb{1}) = \mathbb{1}$ and $R(gh) = R(g)R(h)$. (In other words, R is a group homomorphism from G to $\text{GL}(V)$.) Loosely speaking, we find matrices which represent the symmetry operators.

Examples: representations of the rotation group $SO(3)$

- trivial representation: on scalar quantities $R(\text{Rot}) = \mathbb{1}$, no rotation.
- vector representation: on a vector, $V = \mathbb{R}^3$, $R(\text{Rot}) = \text{Rot}$. This is the so-called *fundamental representation*, in which the rotation matrices take the form that defines the group, namely 3×3 orthogonal matrices with unit determinant.
- tensor representation: on a tensor, $V = \mathbb{R}^3 \times \mathbb{R}^3$, $R(\text{Rot}) = \text{Rot} \otimes \text{Rot}$, because $T_{ij} \mapsto R_i^k R_j^l T_{kl}$.

Similarly, Lie algebras have representations and from a representation of a Lie algebra, we can always construct the associated representation of the group (by using the exponential map). Thus, we will sometimes be a bit sloppy and switch back and forth between algebra and group. Let's make all of this more concrete by looking at the Lorentz and the Poincaré groups as our examples. These are our most important examples, because these encode the fundamental symmetries of spacetime, on which we are constructing our quantum field theory.

2.3 Lorentz transformations and the Lorentz group

Lorentz transformations, abstractly denoted by Λ , act on 4-vectors that denote the spacetime location of an event ³, i.e.,

$$x^\mu = \begin{pmatrix} t \\ \vec{x} \end{pmatrix} \quad (33)$$

as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (34)$$

where the defining equation for a Lorentz transformation is

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\sigma\rho}. \quad (35)$$

This equation says that the Minkowski metric is left invariant under Lorentz transformations, which implies that scalar products built with this metric are invariant under Lorentz transformations. Because η is the Minkowski metric, the Lorentz group is $SO(3, 1)$, and contains boosts and spatial rotations, instead of being $SO(4)$, the group of rotations of 4-dimensional space (which Eq. (35) would define for $\eta_{\mu\nu} \rightarrow \delta_{\mu\nu}$).

From Eq. (35), we have that $\Lambda_\nu{}^\mu = \eta_{\mu\kappa} \eta_{\lambda\nu} \Lambda^\lambda{}_\kappa$ is the inverse Lorentz transformation. This is easiest to see by writing Eq. (35) in matrix notation, where it reads

$$\Lambda^T \eta \Lambda = \eta, \quad (36)$$

where the first Λ is transposed, in order for the index contraction in Eq. (35) to match index contraction for matrix multiplication. From Eq. (36), we then have that

$$\Lambda^{-1} = \eta^{-1} \Lambda^T \eta, \quad (37)$$

which, in index notation, becomes

$$(\Lambda^{-1})^\mu{}_\nu = \eta^{\mu\kappa} (\Lambda^T)_\kappa{}^\lambda \eta_{\lambda\nu} = \eta^{\mu\kappa} \Lambda^\lambda{}_\kappa \eta_{\lambda\nu} = \Lambda_\nu{}^\mu. \quad (38)$$

When acting on 4-vectors, Λ are in their fundamental representation, which you can think of as the representation that is used to define the group. How is the associated Lie algebra $\mathfrak{so}(3, 1)$ characterized?

We use that we can expand the exponential map to first order in the Lie algebra elements, if we consider an infinitesimal transformation. For the fundamental representation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \mathcal{O}(\omega^2) \quad (39)$$

³Note that we use units in which $c = 1$.

for an infinitesimal transformation. Eq. (35) then implies a property of the ω 's:

Mini-Exercise 3. What holds for $\omega^\mu{}_\nu$, such that (35) holds?

Solution.

$$\begin{aligned} (\delta^\mu{}_\rho + \omega^\mu{}_\rho) \eta_{\mu\nu} (\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) &\stackrel{!}{=} \eta_{\sigma\rho} \\ \eta_{\rho\sigma} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + \mathcal{O}(\omega^2) &= \eta_{\sigma\rho} \implies \omega_{\rho\sigma} = -\omega_{\sigma\rho}. \end{aligned}$$

It holds that

$$\omega_{\mu\nu} = -\omega_{\nu\mu}, \quad (40)$$

i.e., $\omega_{\mu\nu}$ is an antisymmetric 4×4 matrix and therefore has 6 independent components that can be nonzero. Depending on which components we choose to be nonzero, we obtain a different group element of the Lorentz group.

Let us consider an example: We choose $\omega^{12} = -\omega^{21} = \theta$ and set all other components of ω to zero. Note that we have to be careful with the upper and lower indices on ω , so there will be an $\eta_{\mu\nu}$ that will make an appearance below. We obtain that

$$\begin{aligned} \Lambda^\mu{}_\nu &= \delta^\mu{}_\nu + \omega^{\mu\rho} \eta_{\rho\nu} \\ &= \mathbb{1} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta & 0 \\ 0 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (41)$$

We observe that this generates nothing but a (infinitesimal) rotation of the four-vector x^μ about the z -axis by an angle θ . We also note that the contraction $\omega^{\mu\rho} \eta_{\rho\nu}$ essentially flips the sign, i.e., $\omega^1{}_2 = -\omega^{12}$ and similarly $\omega^2{}_1 = -\omega^{21}$.

Similarly, if we choose $\omega^{01} = -\omega^{10} = \theta$, we obtain

$$\Lambda^\mu{}_\nu = \mathbb{1} + \begin{pmatrix} 0 & \theta & 0 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (42)$$

which we can recognize as an infinitesimal boost along the x -axis, with θ being the rapidity, $\tanh \theta = v/c$. In this case, we have used that $\omega^0{}_1 = \omega^{01} \eta_{11} = -\omega^{01}$ and $\omega^1{}_0 = \omega^{10} \eta_{00} = -\omega^{10}$.

These examples help us to see that the six entries in $\omega^{\mu\nu}$ which can be nonzero, select, which among the six possible “basis” transformations (3 rotations along the 3 spatial axis, and 3 boosts along these axis), can be performed and by which amount the physical system is rotated and/or boosted. If we choose more than one component of $\omega^{\mu\nu}$ to be non-zero, we get the corresponding combination of these “basis” transformations.

For a general representation $U(\Lambda)$ of the Lorentz transformation Λ , we have that

$$U(\mathbb{1} + \omega) = \mathbb{1} + \frac{i}{2} \omega_{\mu\nu} M^{\mu\nu} + \mathcal{O}(\omega^2). \quad (43)$$

In this expression, the $\omega^{\mu\nu}$ still selects, which transformation is performed and determines the “amount” of the transformation, but the “basis transformations” are now encoded in the $M^{\mu\nu}$. The $M^{\mu\nu}$ are called the *generators* of the Lorentz group, and there are six of them, representing the three independent rotations and three independent boosts. We have that

$$\begin{aligned} J^i &= \frac{1}{2} \epsilon^i_{jk} M^{jk} && \text{generates rotations} \\ K^i &= M^{i0} && \text{generates boosts,} \end{aligned}$$

where ϵ_{ijk} is the Levi-Civita symbol, which is fully antisymmetric under permutations of its indices and $\epsilon_{123} = 1$. Thus, M^{12} , M^{13} and M^{23} generate rotations (along the z -, y - and x -axis, respectively) and M^{01} , M^{02} and M^{03} generate boosts along the x -axis, y -axis and z -axis, respectively.

A Lorentz transformation can act on many different objects, not just on four-vectors. In particular, we will later in the course encounter *spinors*, which are objects that have spinor indices. These are indices, i.e., a spinor is a collection of functions, but they are *not* spacetime indices. Therefore, to have a Lorentz transformation act on a spinor, the $M^{\mu\nu}$ need to carry the appropriate indices, i.e., each of the six $M^{\mu\nu}$ ’s, such as M^{01} , M^{12} etc., must be a matrix with indices in the space that it acts on.

This is somewhat abstract at this moment, so in order to make it less abstract, we consider the case in which the Lorentz transformation acts on a four-vector. We already know that we can write this in the form of Eq. (39), but now we want to understand how to write it in the form Eq. (43), in which the generators appear explicitly. In fact, for the fundamental representation of the Lorentz group, we have that

$$(M_{\mu\nu})_{\kappa\lambda} = -i(\eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\nu\kappa}\eta_{\mu\lambda}). \quad (44)$$

By plugging this into Eq. (43), we get back Eq. (39).

While it seems unnecessarily complicated to introduce the M ’s for the action on 4-vectors, the main point about Eq. (43) is that it is general; it describes the action of a Lorentz transformation on any object.

The *defining property* of the generators of the Lorentz group is that they satisfy a commutation relation. The abstract definition of the Lie algebra of the Lorentz group is through this commutation relation:

The Lie algebra of the Lorentz group $SO(3,1)$, is defined by the commutator relation of its generators, which is

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma}) - i(\eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\sigma} M^{\mu\rho}). \quad (45)$$

You will derive this commutation relation in the exercises. You can think of the Lorentz group as being defined by this commutation relation. When we talk about different elementary particles and different fields, they all arise from thinking about different representations of the Lorentz group, i.e., many properties of elementary particles follow from this commutation relation above. At this stage, this is still a rather abstract notion, but over the course of this course, we will see

the commutation relation Eq. (45) “unfold its power”.

2.4 Poincaré group and why we classify particles by their mass and spin

We classify elementary particles by their mass and spin, plus quantum numbers associated to internal symmetries. For instance, we describe the electron as a particles with rest-mass 511 keV and spin-1/2 (and electric charge -1). Why do we do so? Is it just a conventional choice and we could be using some completely different characteristics? The answer is no. There is a deep mathematical reason and it has to do with the structure of the *Poincaré-group* - a generalization of the Lorentz group - and its so-called *Casimir-operators*.

The Poincaré group is an extension of the Lorentz group which, in addition to boosts and rotations, contains translations, under which $x^\mu \mapsto x^\mu + a^\mu$. A transformation by an element of the Poincaré group can be written as $x^\mu \mapsto \Lambda^\mu_\nu x^\nu + a^\mu$.

This is the full symmetry that 3+1-dimensional Minkowski spacetime enjoys.

An infinitesimal translation in a general representation can be written as

$$U(a) = \mathbb{1} + ia_\mu P^\mu, \quad (46)$$

where P^μ is the generator of translations. By Noethers theorem, P^μ will be identified as the 4-momentum in the corresponding representation. Its commutation relations with the other generators of the Poincaré group are

$$[P^\mu, M^{\rho\sigma}] = i(\eta^{\mu\sigma} P^\rho - \eta^{\mu\rho} P^\sigma) \quad (47)$$

$$[P^\mu, P^\nu] = 0. \quad (48)$$

Now let us consider some state of n particles, which transforms under actions of the Poincaré group. Under such transformations, its properties, such as its 4-momentum, change.

However, the Poincaré group has two *Casimir invariants*. These are (in the simplest case) quadratic combinations of generators, which commute with all other generators. Therefore, their eigenvalues are unchanged under the action of group elements and they provide *invariant* characterizations of particles.

$P^2 = P_\mu P^\mu$ is the first Casimir invariant and $W^2 = W_\mu W^\mu$, with $W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma$ the Pauli-Lubanski-pseudovector, is the second.

Mini-Exercise 4. Show that P^2 commutes with all generators of the Poincaré group.

Solution.

$$\begin{aligned} [P^2, M_{\mu\nu}] &= [P^\rho P_\rho, M_{\mu\nu}] \\ &= P^\rho [P_\rho, M_{\mu\nu}] + [P^\rho, M_{\mu\nu}] P_\rho \\ &= P^\rho (i\eta_{\rho\nu} P_\mu - i\eta_{\rho\mu} P_\nu) + (i\eta_{\rho\nu} P_\mu - i\eta_{\rho\mu} P_\nu) P^\rho \\ &= i(P_\nu P_\mu - P_\mu P_\nu + P_\mu P_\nu - P_\nu P_\mu) \\ &= 0. \end{aligned}$$

P^2 acting on a state with some 4-momentum yields the eigenvalue m^2 , i.e., because P^2 is a Casimir operator of the Poincaré group, we label elementary particles by their rest mass.

But what is the physical meaning of W^2 ?

$$W^2 = W_\mu W^\mu = \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma \epsilon^\mu_{\chi\lambda\tau} M^{\chi\lambda} P^\tau. \quad (49)$$

Let's consider this in a massive particles rest frame (massless particles are a separate case and we will get to them later).

Then $P \rightarrow (m, \vec{0})$ and $W^0 = 0$. This holds, because ϵ is totally antisymmetric and because the only non-zero component of P is P^0 .

$$W^i = \frac{1}{2} \epsilon^i_{\mu\nu 0} M^{\mu\nu} P^0, \quad (50)$$

here μ, ν must be spatial indices, but $\neq i$. Therefore, $\epsilon^i_{jk0} = \epsilon^i_{jk}$, the 3d Levi-Civita symbol. Thus,

$$W^i = J^i P^0 = m J^i \implies W^2 = -m^2 \vec{J} \cdot \vec{J}. \quad (51)$$

Now we need to interpret which angular momentum it is that shows up here. Which angular momentum does an elementary particle have? *Intrinsic* angular momentum, i.e., spin. You might remember from QM, that the eigenvalues of $\vec{J}^2$ are $s(s+1)$, with s the spin.

$\implies$ Because W^2 is the 2nd Casimir operator of the Poincaré group, we label massive elementary particles by their spin.

We have thus come to our first concrete result from our more abstract consideration of group theory:

We have learned that there is a reason why we label elementary particles by mass and spin. This is not an arbitrary choice, but a direct consequence of the fundamental symmetry-structure of Minkowski spacetime and the properties of the underlying Poincaré group.

Next, we may wonder, what spin values⁴ are allowed? Can we have elementary particles with spin 0? spin 1/2? Spin 1? What about non-half-integers? Is there a particle with spin 2/3? or spin M ? To figure this out, we will classify the representations of the Lorentz group. This will determine what type of fields we will focus on for the rest of the course.⁵ Generally, for a field with a general Lorentz index A (could be a 4-vector index, or two 4-vector indices, such that the field is a tensor, but we'll also encounter spinor indices, which label the components of a spinor, but are *not* spacetime indices), $\phi_a(x)$, we have

$$\phi'_a(x) = L_a{}^b(\Lambda) \phi_b(\Lambda^{-1}x). \quad (52)$$

⁴All in units of $\hbar$, which we set to 1.

⁵In the current discussion, we are switching back and forth between considering particles and fields. In this, we are already using a result that we will see a little later in the course, namely that elementary particles show up as excitations of fields. Therefore, it is to some extent equivalent to talk about particles or about the associated fields, because the properties of the particles follow from the properties of the fields. However, let us highlight that there is a difference when it comes to representations of the Poincaré group: fields transform in the finite-dimensional representations of the group, i.e., they are constructed from a finite set of components. In contrast, particles transform in the infinite-dimensional representation of the Poincaré group. Physically, this is, loosely speaking because if you have a particle with some four-momentum p^μ , then there are infinitely many other four-momenta p'^μ that are related to p^μ by a boost. The choice of an infinite-dimensional representation is also necessary, because no finite-dimensional representation is unitary, and we would like to have probabilities (or scalar products of a state with itself) to be preserved under Poincaré transformations. Therefore, the representation that a field transforms in is not the same one as the particles that it gives rise to transform in. However, for our purposes at the present, we do not yet need to know this, as we will now simply focus on the representations that the fields can transform in.

The matrices $L_a{}^b(\Lambda)$ form a *representation* of the Lorentz group, i.e.,

$$L_a{}^b(1 + \omega) = \delta_a{}^b + \frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})_a{}^b \quad (53)$$

where $(M^{\mu\nu})_a{}^b$ are representation matrices of the $\mathfrak{so}(3,1)$ Lie algebra, so that

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma}) - i(\eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\sigma} M^{\mu\rho}). \quad (54)$$

To understand which spins elementary particles can have, we must find all possible (finite-dimensional) matrices $M_{ab}^{\mu\nu}$ that obey these commutation relations, in order to find the possible fields that we can write down. This sounds like a challenging problem, but it turns out that we are lucky if we know something about the representation of the Lie algebra $SU(2)$ ⁶.

From QM, we know that $[J_i, J_j] = i\epsilon_{ijk}J_k$, which is the $SU(2)$ Lie algebra, is satisfied by sets of 3 hermitian matrices of size $(2j+1) \times (2j+1)$, where the eigenvalues of J_3 are $-j, -j+1, \dots, +j$. (If you would like a “refresher” on this, a good place to read up on it is, e.g., Sakurai “Modern Quantum Mechanics”.)

Our luck lies in the fact that upon introducing

$$N_i = \frac{1}{2}(J_i - iK_i) \quad (\text{remember : } J_i = \frac{1}{2}\epsilon_{ijk}M_{jk} \text{ and } K_i = M_{i0}) \quad (55)$$

and

$$M_i = \frac{1}{2}(J_i + iK_i) \quad (56)$$

(Note: J_i, K_i are hermitian; N_i is not; in fact $M_i = N_i^\dagger$.) we find that

$$[N_i, N_j] = i\epsilon_{ijk}N_k, \quad [M_i, M_j] = i\epsilon_{ijk}M_k, \quad [N_i, M_j] = 0. \quad (57)$$

The Lie algebra of $SO(3,1)$ is nothing but two separate $SU(2)$ Lie algebras!

Thus, we can build the representations of the $SO(3,1)$ Lie algebra from representations of the $SU(2)$ Lie algebra!

$\implies$ Each irreducible (i.e., not give by a product of two smaller representations) representation of the $SO(3,1)$ Lie algebra is specified by *two* integers or half-integers n' and n , which are the eigenvalues of M_3 and N_3 .

We *label* these representations by n and n' or by the number of components in each representation, $(2n+1)$ and $(2n'+1)$.

To understand the corresponding spin of the field (and the particles that are the excitations of the field), we go back to the Pauli-Lubanski pseudovector and the associated Casimir operator, in the rest-frame, $W^2 = -m^2 \vec{J} \cdot \vec{J}$ and also use that $J_i = M_i + N_i$. Thus,

⁶Note that the Lie algebras for $SO(3)$ and $SU(2)$ are identical. For the groups, there are some subtle differences, which need not directly concern us.

(n, n')	$(2n + 1, 2n' + 1)$	spin	name of the field
$(0, 0)$	$(1, 1)$	0	scalar (singlet)
$(\frac{1}{2}, 0)$	$(2, 1)$	$\frac{1}{2}$	left-handed spinor
$(0, \frac{1}{2})$	$(1, 2)$	$\frac{1}{2}$	right-handed spinor
$(\frac{1}{2}, \frac{1}{2})$	$(2, 2)$	1	vector (this has $2 \cdot 2 = 4$ components, which is the right number for a 4-vector)

Therefore, we now have a clear idea which fields we are going to consider. Rather than guessing that maybe there could be elementary particles with spin $2/3$ (or other non half-integer values) out there, and somehow trying to come up with ideas for what the corresponding fields could be, we already know that such fields/particles do not exist and we do not need to spend our time trying to find a description for them, because our considerations, based on symmetries, tell us that such an effort is futile.

To sum up, by considering the fundamental symmetry of Minkowski spacetime, that a theory of fields and associated particles living on that spacetime has to satisfy, we have developed a comprehensive list of possible fields that can exist. Thus, rather than proceeding by trial-and-error, we have found a systematic structure that the rest of this course (and Quantum Field Theory) will follow. This structure is very restrictive and only allows us to consider fields which are associated to integer or half-integer spins. It is therefore not an accident that all elementary particles have integer or half-integer spin; there are no other options for them, based on the underlying symmetry group, the Poincaré group.

We will work our way through the spin 0, $1/2$ and 1 cases in the course, because, as it turns out, they are all part of the Standard Model of particle physics.

Higher spins ($3/2$, 2) do not correspond to detected elementary particles, although spin $3/2$ plays a role in supergravity, where a spin $3/2$ particle is the superpartner of the graviton. The graviton, which is the expected quantum of the gravitational field, has spin 2.

2.5 Noether's theorem

Symmetries not only help us to understand the building blocks of our theory (i.e., which fields there may be and how we characterize particles), they also imply conserved quantities and thus determine dynamical processes. The link between symmetries and conserved quantities is at the heart of Noether's theorem, just as in classical mechanics.

Noether's theorem in QFT states that:
Every continuous symmetry of the action implies a conserved current density and a conserved charge.

This is similar to Noether's theorem in classical mechanics with the key difference being the conservation of the current. To derive the theorem, we will consider a scalar field; the theorem generalizes to non-zero spin fields, such as the gauge field and spinor fields.

As an example of a continuous spacetime symmetry, consider a translation $x \rightarrow x' = x + d$. How does $\phi \rightarrow \phi'$ look like? Note that we will take the *active point of view*, where we are assuming that the physical field configuration changes (in contrast to the *passive point of view*, where the coordinates change). It should hold that the transformed field at the transformed point is equal

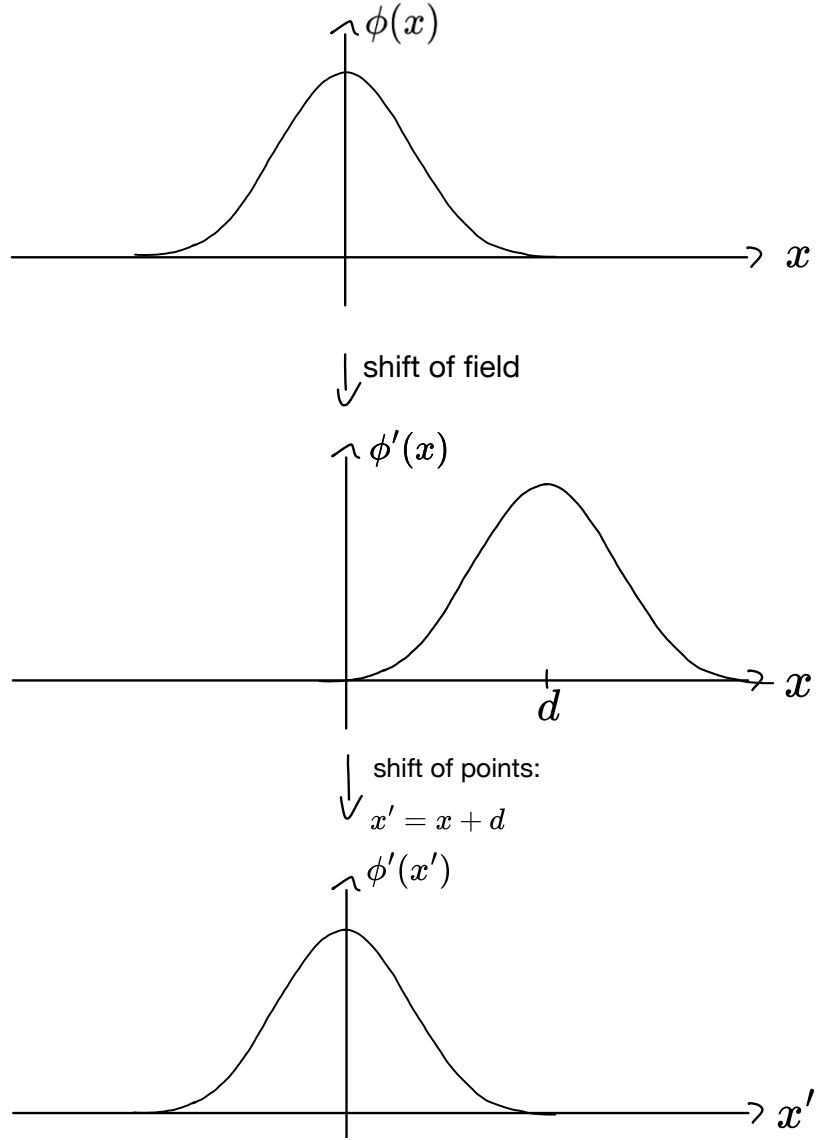


Figure 1: We show a field configuration $\phi(x)$ in the upper panel. In the central panel, we have shifted the field by a distance d (active transformation) and in the lower panel, we have then additionally shifted the coordinates by the same distance d , so that $x' = x + d$.

to the untransformed field at the original point, because, if we are shifting the field, but then also shift all points, the system remains unchanged. Thus

$$\phi'(x') = \phi(x), \quad (58)$$

which is shown in Fig. 1.

Thus, $\phi'(x)$ is defined by applying the inverse transformation to the argument, i.e.,

$$\phi'(x) = \phi(x - d). \quad (59)$$

When we generalize to a Lorentz transformation $x' = \Lambda x$, we have the same behavior: the scalar

field is evaluated at a point that corresponds to the inverse of the transformation.

$$\phi'(x) = \phi(\Lambda^{-1}x). \quad (60)$$

We can also consider *internal* symmetries, e.g., for a complex scalar field $\phi(x)$ taking values in $\mathbb{C}$ instead of in $\mathbb{R}$, we can write an action that has a $U(1)$ symmetry:

$$S_{U(1) \text{ complex scalar}} = \int d^4x \left(\frac{1}{2} \partial_\mu \phi^* \partial^\mu \phi - \frac{1}{2} m^2 \phi^* \phi \right), \quad (61)$$

which is invariant under $\phi(x) \rightarrow e^{i\alpha} \phi(x)$, and, accordingly $\phi^*(x) \rightarrow e^{-i\alpha} \phi^*(x)$. The infinitesimal version of this transformation is

$$\phi'(x) = \phi(x) (1 + i\alpha + \dots). \quad (62)$$

We will consider this example in much more detail later in the lecture.

To derive Noether's theorem, we assume some continuous symmetry, but we do not need to specify whether it is a spacetime symmetry or an internal symmetry. Noether's theorem holds for both. Because we are assuming a continuous symmetry, there is an infinitesimal version of this transformation of the field

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \varepsilon \chi(x). \quad (63)$$

(For a discrete symmetry, there are only finite transformations, e.g. a $\mathbb{Z}_2$ -symmetry under which $\phi(x) \rightarrow -\phi(x)$ has no infinitesimal version. This is why all that follows holds for continuous, but not for discrete symmetries.)

For instance, for an infinitesimal translation, we can write the right-hand side in terms of a Taylor expansion

$$\phi'(x) = \phi(x) + \frac{\partial \phi}{\partial x^\mu} d^\mu + \dots, \quad (64)$$

$$= \phi(x) + \epsilon_\mu \chi^\mu(x), \quad (65)$$

where we consider d^μ to be an infinitesimal shift and we defined $\partial_\mu \phi d^\mu = \epsilon_\mu \chi^\mu(x)$. When we perform a translation in a single direction in spacetime, this reduces back to the form $\epsilon_\mu \chi^\mu \rightarrow \epsilon \chi$. We denote the difference between the transformed and the untransformed field

$$\delta_\varepsilon \phi := \phi' - \phi. \quad (66)$$

Under this change in the field, the Lagrangian changes as follows:

$$\delta_\varepsilon \mathcal{L} = \mathcal{L}' - \mathcal{L} = \mathcal{L}(\phi', \partial \phi') - \mathcal{L}(\phi, \partial \phi) \quad (67)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi} \delta_\varepsilon \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta_\varepsilon \partial_\mu \phi, \quad (68)$$

where $\delta_\varepsilon \partial_\mu \phi = \partial_\mu \phi' - \partial_\mu \phi$. (Note that we're slightly abusing naming conventions, as advertised, because this is the Lagrangian *density*, but we are referring to it as the Lagrangian. This is very common practise in QFT.)

Because we assume that the transformation corresponds to a symmetry of the action, the action

must stay invariant under it. Thus, the Lagrangian may at most change by a total derivative, so we can write

$$\delta_\varepsilon \mathcal{L} = \varepsilon \partial_\mu F^\mu(\phi, \partial\phi, \partial^2\phi, x), \quad (69)$$

where, depending on the symmetry F^μ may actually be zero, so that even the Lagrangian is invariant under the symmetry.

We know that $\delta_\varepsilon \mathcal{L} \sim \varepsilon$, because $\delta_\varepsilon \mathcal{L} \rightarrow 0$ for $\varepsilon \rightarrow 0$. In principle, $F^\mu = F^\mu(\phi, \partial\phi, \partial^2\phi, x)$ can have dependencies on x and on $\partial^2\phi$, etc., even if $\mathcal{L}$ does not.

Now we want to derive the conserved current. $\partial_\mu F^\mu$ is a good starting point, because it already has the required form for a conservation law, $\partial_\mu j^\mu = 0$.

$$\varepsilon \partial_\mu F^\mu = \delta_\varepsilon \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta_\varepsilon \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \partial_\mu \phi. \quad (70)$$

In the next step we use the equations of motion,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0, \quad (71)$$

to rewrite the 1st term into a form that also has a partial derivative in front, as needed to derive a conservation law. Note that this will mean that everything that follows only applies for field configurations which satisfy the equations of motion. (In QFT, these are often called “on-shell” configurations. In a few weeks, when we talk about the path integral quantization, we will explicitly see the difference to the “off-shell” configurations.) We obtain

$$\implies \varepsilon \partial_\mu F^\mu = \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta_\varepsilon \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \partial_\mu \phi \quad (72)$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \phi \right). \quad (73)$$

Thus,

$$\partial_\mu \left(\underbrace{F^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \chi}_{j^\mu} \right) = 0. \quad (74)$$

j^μ is a conserved current.

Example: Energy-momentum tensor and its conservation

In classical mechanics, the symmetry-transformation underlying energy-momentum conservation is a space-time-translation:

$$x^\mu \rightarrow x'^\mu = x^\mu - \epsilon^\mu. \quad (75)$$

(These are really 4 symmetries packaged into one.)

The resulting transformation of the field is, as we wrote above,

$$\phi'(x) = \phi(x + \varepsilon) \quad (76)$$

$$\implies \delta_\varepsilon \phi = \phi(x + \varepsilon) - \phi(x) = \varepsilon^\nu \underbrace{\partial_\nu \phi(x)}_{\chi_\nu} \quad (77)$$

Eq. (76) means that the new field at x is the same as the old one at $x + \varepsilon$, because the shift is by $-\varepsilon$, and we are again using the active view on transformations. In Eq. (77), χ has an index, because there are 4 symmetries. Eq. (77) is to 1st order in ε , because we can Taylor expand $\phi(x + \varepsilon)$. Thus, if we focus on the dependence of $\mathcal{L}$ on x (through its dependence on ϕ),

$$\mathcal{L}'(x) = \mathcal{L}(x + \varepsilon) \quad (78)$$

$$\implies \delta_\varepsilon \mathcal{L} = \mathcal{L}(x + \varepsilon) - \mathcal{L}(x) \quad (79)$$

$$= \varepsilon^\mu \partial_\mu \mathcal{L}(x) \quad (\text{to 1st order in } \varepsilon) \quad (80)$$

$$= \varepsilon^\nu \partial_\mu \underbrace{(\delta^\mu_\nu \mathcal{L})}_{:= F^\mu_\nu} \quad (81)$$

Now we can use the general expression we derived before to get the conserved currents. Because we are looking at 4 symmetries at the same time, we will have 4 conserved currents, each of which is a 4-vector. In Eq. (81), you can think of the index μ as the index that belongs to a conserved current (which is a four-vector) and the index ν as the one that labels the four distinct currents that there are for the four distinct translations. Which translation is performed, is selected by the non-zero components of ε^ν .

To “package” the four conserved currents into one expression, we write

$$j^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \chi_\nu - F^\mu_\nu \quad (82)$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}. \quad (83)$$

This conserved *tensor* is usually written as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}. \quad (84)$$

It is conserved, $\partial_\mu T^{\mu\nu} = 0$, and called the energy-momentum (or stress-energy) tensor.

Let us clarify the status of $T^{\mu\nu}$: we could derive the expression $j^{\mu\nu}$ in Eq. (83) from spacetime translations even if spacetime translations would not correspond to a symmetry of the action. However, we would *not* have that $j^{\mu\nu}$ is a conserved quantity. In other words, if $S' \neq S$ under a transformation, we will have $\partial_\mu j^\mu \neq 0$ for the j^μ corresponding to this transformation.

Side-note: The energy-momentum tensor is an entry-point into General Relativity, because, if we promote $\eta^{\mu\nu} \rightarrow g^{\mu\nu}$, then $T^{\mu\nu}$ acts as a source for spacetime curvature in the Einstein equations. The physical meaning behind that is that any form of energy or momentum sources spacetime curvature.

In our statement of Noether’s theorem, we also mentioned the conservation of a charge. Let us derive what the conserved charge is. From the conservation of the current, we can also derive the conservation of a charge:

$$Q(t) = \int d^3x j^0(t, \vec{x}). \quad (85)$$

It holds that

$$\dot{Q} = \frac{d}{dt} Q(t) = 0, \quad (86)$$

if we assume that all fields and their derivatives vanish at $|x| \rightarrow \infty$, i.e., we only consider nonzero field configurations away from spatial infinity. This is reasonable to describe all realistic physical situations that we are interested in (e.g., particle physics experiments at CERN, phonons in the Bose-Einstein-Condensates of our experimental friends in Neuenheimer Feld, or superconducting Cooper-pairs in superconductors in various labs, all of which are described by QFT.)

We can show $\dot{Q} = 0$ as follows:

$$\dot{Q} = \frac{d}{dt} \int d^3x j^0(t, \vec{x}) = \int d^3x (\partial_0 j^0(t, \vec{x})) \quad (87)$$

$$= - \int d^3x \partial_i j^i(t, \vec{x}) \quad (\text{by conservation of the current}) \quad (88)$$

$$= - \int dxdydz (\partial_x j^x + \partial_y j^y + \partial_z j^z) \quad (89)$$

$$= - \int dydz j^x \Big|_{x \rightarrow \pm\infty} + \int dx dz j^y \Big|_{y \rightarrow \pm\infty} + \int dx dy j^z \Big|_{z \rightarrow \pm\infty} \quad (90)$$

$$= 0, \quad (91)$$

if fields and derivatives vanish at $|x| \rightarrow \infty$, so that j vanishes there.

Let us highlight that the conservation of a *current* is stronger than the conservation of the charge, because it implies that the charge is conserved *locally*, i.e., changes of the charge in a (finite) volume in time must be accounted for by a current flowing through the surface of the volume. To see this, write:

$$\frac{dQ_V}{dt} = - \int_V d^3x \vec{\nabla} \cdot \vec{j} = - \int_{A=\partial V} \vec{j} \cdot d\vec{S}. \quad (92)$$

Q_V is the charge in a volume V . In the last step we used Gauss' theorem for volume integrals of divergences.

Example: the conserved charges following from the conservation of the energy-momentum tensor are:

$$\int d^3x T^{00} = \underbrace{\int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \right)}_{\substack{\text{We recognize this} \\ \text{as the Hamiltonian}}} = H =: P^0 \quad (93)$$

The other conserved charges are the spatial momenta, so $P^\nu = \int d^3x T^{0\nu}$ is conserved.

Noether's theorem also applies to continuous internal symmetries. We'll consider an example later in the course.

Mini-Exercise 5. Take

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \quad (94)$$

What is P^i ?

Solution.

$$\begin{aligned} P^i &= \int d^3x T^{0i} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \partial^i \phi - \underbrace{\eta^{0i}}_{=0} \mathcal{L} \\ &= \dot{\phi} \partial^i \phi. \quad (\text{not surprisingly only dependent on kinetic energy}) \end{aligned}$$

We note that the conserved quantities in turn are the *generators* of the associated symmetry. This closes our considerations of symmetries. We have learned that symmetries are encoded in groups. Continuous symmetries of interest in physics are Lie groups, for which each symmetry transformation can be generated by the generators of the Lie algebra. In turn, Noether's theorem tells us that each symmetry leads to a conserved quantity. This conserved quantity is the generator that generates this symmetry.

If we did not know about the Lie group associated to a symmetry, we could therefore learn about it from the action of the symmetry and the resulting conserved quantities.

3 Canonical quantization of the free scalar field

We are now ready to quantize the scalar field. We will first quantize it according to canonical quantization, because that makes the connection between fields and particles clear. We will then quantize it according to path-integral quantization, because that makes the quantum superposition principle clear. By introducing both canonical and path-integral formalism early on (as not all QFT books do), we are also acquiring a versatile toolbox with which to build more complicated QFTs (e.g., the Standard Model of particle physics) later. In addition, the canonical framework and the path-integral framework provide us with a different type of intuition about quantum fields, so it is very useful to know both.

3.1 Canonical Quantization of the real scalar field in the Schrödinger picture

We start with the non-interacting field, $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. In the Schrödinger picture, we only consider the spatial dependence of the field (the time-dependence will be carried by the states).

We work in a formal analogy to quantum mechanics, where we generalize its formal structure. In Quantum Mechanics, we have the operators x and $p = \frac{dL}{dx}$, for which we impose $[x, p] = i$ (in units where $\hbar = 1$). We will generalize this structure by making the analogy $x \simeq \phi(\vec{x})$ and $p \simeq \Pi(\vec{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$.⁷

In analogy to quantum mechanics, we promote the field and its conjugate field to operators⁸ and demand commutation relations:

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\delta^3(\vec{x} - \vec{y}) \quad (95)$$

$$[\phi(\vec{x}), \phi(\vec{y})] = 0 = [\Pi(\vec{x}), \Pi(\vec{y})]. \quad (96)$$

The operators have no time dependence, because we work in the Schrödinger picture, where states carry time-dependence and operators do not. The operators depend on the spatial position, $\vec{x}$, because they are fields and thus describe infinitely many degrees of freedom (one at each spacetime point) rather than finitely many, as in QM.

So far, these are formal expressions. How can we build the Hilbert space of states and what operators will the states be eigenstates of? Also, how does the field act on the states; what is the physical meaning of that?

To answer all of these questions, we start from the observation that the Hamiltonian H is very reminiscent of an (infinite) set of harmonic oscillators (one for each point) and we know how to quantize the harmonic oscillator. In detail, the Hamiltonian of the free scalar field is:

$$H = \frac{1}{2} \int d^3x (\Pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2) \quad (97)$$

In comparison, the Hamiltonian of the harmonic oscillator in quantum mechanics is $H_{\text{QM}} = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2$.

⁷Note that $\Pi(\vec{x})$ has nothing to do with the physical momentum, i.e., the momentum $\vec{P}$ that we can derive as a conserved quantity from Noether's theorem for spatial translation. The physical momentum can be expressed in terms of the fields, as is done at the end of the last chapter.

⁸Note: We are *not* putting hats on operators. It should be clear from the context, when we are dealing with the QM theory in its canonical formulation and hence with operators.

However, the structure of H is not fully analogous to H_{QM} , because the term $\sim (\vec{\nabla}\phi)^2$ couples harmonic oscillators at neighboring points.

We can actually decouple the oscillators by going to Fourier space, where $(\vec{\nabla}\phi)^2$ will become $\sim (\vec{p}\phi)^2$ and H will become an infinite set of *decoupled* harmonic oscillators, one for each Fourier mode (or momentum).

We write the field $\phi(\vec{x})$ in terms of its Fourier modes $\tilde{\phi}(\vec{p})$ as

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \tilde{\phi}(\vec{p}) \quad (98)$$

and the inverse relation

$$\tilde{\phi}(\vec{p}) = \int d^3x e^{-i\vec{p}\cdot\vec{x}} \phi(\vec{x}), \quad (99)$$

where, to show this inverse relation, it is crucial that

$$\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^3(\vec{x}-\vec{y}). \quad (100)$$

You will prove this representation of the Dirac delta-distribution in the exercises.

Similar expressions hold for $\Pi(\vec{x})$. From these, and the commutators for $\phi(\vec{x})$ and $\Pi(\vec{y})$, we can deduce $[\tilde{\phi}(\vec{p}), \tilde{\Pi}(\vec{q})]$.

Mini-Exercise 6. Deduce what the commutator $[\tilde{\phi}(\vec{p}), \tilde{\Pi}(\vec{q})]$ is.

Solution.

$$\begin{aligned} [\tilde{\phi}(\vec{p}), \tilde{\Pi}(\vec{q})] &= \int d^3x \int d^3y e^{-i\vec{p}\cdot\vec{x}} e^{-i\vec{q}\cdot\vec{y}} \underbrace{[\phi(\vec{x}), \Pi(\vec{y})]}_{i\delta^3(\vec{x}-\vec{y})} \\ &= i \int d^3x e^{-i\vec{x}\cdot(\vec{p}+\vec{q})} \\ &= i(2\pi)^3 \delta^3(\vec{p}+\vec{q}). \end{aligned}$$

Further, one can similarly check that

$$[\tilde{\phi}(\vec{p}), \tilde{\phi}(\vec{q})] = 0 = [\tilde{\Pi}(\vec{p}), \tilde{\Pi}(\vec{q})]. \quad (101)$$

To evaluate H in Fourier space, let's first focus on the term

$$\int d^3x (\vec{\nabla}\phi)^2 = \int d^3x \int \frac{d^3p}{(2\pi)^3} i\vec{p} e^{i\vec{p}\cdot\vec{x}} \tilde{\phi}(\vec{p}) \int \frac{d^3q}{(2\pi)^3} i\vec{q} e^{i\vec{q}\cdot\vec{x}} \tilde{\phi}(\vec{q}). \quad (102)$$

The scalar product between the two gradients, $\vec{\nabla} \cdot \vec{\nabla}$ becomes the scalar product between $\vec{p}$ and $\vec{q}$. We proceed by rearranging the terms so that all those involving x are grouped together in the

back:

$$\int d^3x (\vec{\nabla}\phi)^2 = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} (-\vec{p} \cdot \vec{q}) \tilde{\phi}(\vec{p}) \tilde{\phi}(\vec{q}) \underbrace{\int d^3x e^{i\vec{p} \cdot \vec{x}} e^{i\vec{q} \cdot \vec{x}}}_{(2\pi)^3 \delta^3(\vec{p} + \vec{q})} \quad (103)$$

$$= \int \frac{d^3p}{(2\pi)^3} (+\vec{p}^2) \tilde{\phi}(\vec{p}) \tilde{\phi}(-\vec{p}). \quad (104)$$

$\tilde{\phi}(-\vec{p})$ is related to $\tilde{\phi}(\vec{p})$ through complex conjugation, because $\phi(\vec{x})$ is real:

$$\phi(\vec{x}) = \phi^*(\vec{x}) \implies \tilde{\phi}^\dagger(\vec{p}) = \tilde{\phi}(-\vec{p}). \quad (105)$$

We write a dagger, because we are dealing with operators. The classical field satisfies $\phi^*(\vec{p}) = \phi(-\vec{p})$. Overall, we arrive at

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left(\underbrace{|\tilde{\Pi}|^2}_{\tilde{\Pi} \cdot \tilde{\Pi}^\dagger} + \underbrace{(\vec{p}^2 + m^2)}_{=\omega_{\vec{p}}^2} |\tilde{\phi}|^2 \right). \quad (106)$$

Now we have achieved full analogy with an (infinite, but not coupled) set of harmonic oscillators and so we know how to proceed to construct the Hilbert space!

At a conceptual level, the expression highlights

- the difference to QM: one (or finitely many) vs. infinitely many degrees of freedom
- a dangerous assumption: We assumed in this derivation, that this theory is consistent up to arbitrarily high energies (arbitrarily high $\omega_{\vec{p}}$). We will recap the consequences when we come to the topic of Ultraviolet (UV) divergences later.

Note: UV has nothing to do with a frequency of light here; in QFT, it refers to “high energy”.

3.2 Creation and annihilation operators and construction of the Fock space

We have rewritten the Hamiltonian for the non-interacting scalar field as an infinite set of decoupled harmonic oscillators, one for each momentum $\vec{p}$.

Conceptually, this highlights the difference between QM and QFT: finitely vs. infinitely many degrees of freedom (we will see how they are related to particles).

Mathematically, it means that we can follow the procedure to solve the simple harmonic oscillator in QM and generalize to QFT.

For the simple harmonic oscillator in QM, with Hamiltonian $H_{\text{QM}} = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2$, we introduce $a = \frac{1}{2}(\sqrt{2\omega m}x + i\sqrt{\frac{2}{\omega m}}p)$ and obtain $H = \omega(a^\dagger a + \frac{1}{2})$ and $[a, a^\dagger] = 1$ and can construct all eigenstates of H with the “ladder” operators $a, a^\dagger$.

In analogy, we define an operators a for each $\vec{p}$, i.e., one simple harmonic oscillator for each of the infinitely many values of the spatial momentum $\vec{p}$, and check whether this works out. (Note that this is not a priori clear, because $\tilde{\phi}(\vec{p})$ is conjugate to $\tilde{\Pi}(-\vec{p})$ and not $\tilde{\Pi}(\vec{p})$, because $[\tilde{\phi}(\vec{p}), \tilde{\Pi}(-\vec{q})] =$

$i(2\pi)^3\delta^3(\vec{p}-\vec{q})$.) We define:

$$a_{\vec{p}} = \frac{1}{2} \left(\sqrt{2\omega_{\vec{p}}} \tilde{\phi}(\vec{p}) + i\sqrt{\frac{2}{\omega_{\vec{p}}}} \tilde{\Pi}(\vec{p}) \right) \quad (107)$$

The form of $a_{\vec{p}}^\dagger$ follows:

$$a_{\vec{p}}^\dagger = \frac{1}{2} \left(\sqrt{2\omega_{\vec{p}}} \tilde{\phi}(-\vec{p}) - i\sqrt{\frac{2}{\omega_{\vec{p}}}} \tilde{\Pi}(-\vec{p}) \right), \quad (108)$$

where we used that $\tilde{\phi}^\dagger(\vec{p}) = \tilde{\phi}(-\vec{p})$ and $\tilde{\Pi}^\dagger(\vec{p}) = \tilde{\Pi}(-\vec{p})$. In the next step, we have to figure out the commutation relations of $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ from those for the field and its conjugate field.

Mini-Exercise 7. What is $[a_{\vec{p}}, a_{\vec{q}}^\dagger]$? Work efficiently and don't write out terms that will be zero.

Solution. By using $a_{\vec{p}}$ expressed through $\tilde{\phi}$, $\tilde{\Pi}$, we can rewrite $[a_{\vec{p}}, a_{\vec{q}}^\dagger]$ in terms of the commutators of $\tilde{\phi}$ and $\tilde{\Pi}$. By using that $[\tilde{\phi}(\vec{p}), \tilde{\phi}(\vec{q})] = 0 = [\tilde{\Pi}(\vec{p}), \tilde{\Pi}(\vec{q})]$, we can directly reduce this to

$$\begin{aligned} [a_{\vec{p}}, a_{\vec{q}}^\dagger] &= \left[\frac{1}{2} \sqrt{2\omega_{\vec{p}}} \tilde{\phi}(\vec{p}), \frac{-i}{2} \sqrt{\frac{2}{\omega_{\vec{q}}}} \tilde{\Pi}(-\vec{q}) \right] + \left[\frac{1}{2} i \sqrt{\frac{2}{\omega_{\vec{p}}}} \tilde{\Pi}(\vec{p}), \frac{1}{2} \sqrt{2\omega_{\vec{q}}} \tilde{\phi}(-\vec{q}) \right] \\ &= \frac{-i}{2} \sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}} i(2\pi)^3 \delta^3(\vec{p}-\vec{q}) + \frac{i}{2} \sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} i(2\pi)^3 (-) \delta^3(-\vec{q}+\vec{p}) \\ &= (2\pi)^3 \delta^3(\vec{p}-\vec{q}). \end{aligned}$$

In the last step, we set $\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}} \delta^3(\vec{p}-\vec{q}) = \delta^3(\vec{p}-\vec{q})$, because, wherever the prefactor $\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}} \neq 1$, $\delta^3(\vec{p}-\vec{q}) \rightarrow 0$.

Similarly, we can derive that

$$[a_{\vec{p}}, a_{\vec{q}}] = 0 = [a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger]. \quad (109)$$

Thus, we have established analogous commutations relations between the a 's and $a^\dagger$'s as those in QM. Now, we want to check whether we can also rewrite the Hamiltonian in a way that is analogous to QM. For that, it is useful to write $\tilde{\phi}(\vec{p})$ and $\tilde{\Pi}(\vec{p})$ in terms of $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$.

From (107) and (108), we get

$$\tilde{\phi}(\vec{p}) = \frac{1}{\sqrt{2\omega_{\vec{p}}}} (a_{\vec{p}} + a_{-\vec{p}}^\dagger) \quad (110)$$

$$\tilde{\Pi}(\vec{p}) = -i\sqrt{\frac{\omega_{\vec{p}}}{2}} (a_{\vec{p}} - a_{-\vec{p}}^\dagger). \quad (111)$$

Thus, the Hamiltonian becomes

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left(|\tilde{\Pi}|^2 + \omega_{\vec{p}}^2 |\tilde{\phi}|^2 \right) \quad (112)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left(\frac{-i\omega_{\vec{p}}}{2} (a_{\vec{p}} - a_{-\vec{p}}^\dagger) i (a_{\vec{p}}^\dagger - a_{-\vec{p}}) + \omega_{\vec{p}}^2 \frac{1}{2\omega_{\vec{p}}} (a_{\vec{p}} + a_{\vec{p}}^\dagger) (a_{\vec{p}}^\dagger + a_{-\vec{p}}) \right) \quad (113)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\vec{p}}}{4} \left(\cancel{a_{\vec{p}} a_{\vec{p}}^\dagger} - \cancel{a_{\vec{p}} a_{-\vec{p}}^\dagger} + \cancel{a_{-\vec{p}}^\dagger a_{\vec{p}}^\dagger} + \cancel{a_{-\vec{p}}^\dagger a_{-\vec{p}}} + \cancel{a_{\vec{p}} a_{\vec{p}}^\dagger} + \cancel{a_{\vec{p}} a_{-\vec{p}}^\dagger} + \cancel{a_{-\vec{p}}^\dagger a_{\vec{p}}^\dagger} + \cancel{a_{-\vec{p}}^\dagger a_{-\vec{p}}} \right) \quad (114)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\vec{p}}}{2} (a_{\vec{p}} a_{\vec{p}}^\dagger + \underline{a_{-\vec{p}}^\dagger a_{-\vec{p}}}). \quad (115)$$

In the underlined term, we rename the integration variable $\vec{p} \rightarrow -\vec{p}$. Under this change, $\omega_{\vec{p}} = \omega_{-\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ and $\int d^3p \rightarrow \int d^3p$. Thus,

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\vec{p}}}{2} (a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}}). \quad (116)$$

Just like in QM, it will be useful to rewrite the order of $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$, which we can do by using the commutator.

$$\Rightarrow H = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a_{\vec{p}}^\dagger a_{\vec{p}} + \frac{1}{2} \underbrace{\left[a_{\vec{p}}, a_{\vec{p}}^\dagger \right]}_{(2\pi)^3 \delta^3(0)} \right). \quad (117)$$

We interpret $(2\pi)^3 \delta^3(0) = \int d^3x e^{i\vec{0} \cdot \vec{x}} = \int d^3x = \text{Vol}(\mathbb{R}^3) =: V$ as the volume of space, so that

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a_{\vec{p}}^\dagger a_{\vec{p}} + \frac{V}{2} \right). \quad (118)$$

The part $\int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \frac{V}{2}$ is *divergent*, due to the contribution of zero-point energies of harmonic oscillators with arbitrarily high frequency (even if V is kept finite). This is called an ultraviolet (UV) divergence.

We will encounter more UV divergences soon, when we will discuss *regularization* and *renormalization*. The physical reason behind UV divergences is always that we consider momentum integrals to arbitrarily high momenta. In doing so, we are assuming that the theory that we are considering holds to arbitrarily small distances (high momenta). This is clearly a (wild) extrapolation, because experimentally we can only check QFTs (e.g., those that make up the Standard Model) down to $\sim 10^{-19}\text{m}$ (or 10TeV). In QFTs relevant to condensed matter, there is a known UV cutoff in the form of the lattice spacing between atoms. Below this cutoff, it does not make sense to think about, e.g., phonons or other collective excitations that are described by a QFT.

Specifically, the UV divergence associated to the *vacuum energy*, $V \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\vec{p}}}{2}$, can be ignored in QFT on $\mathbb{M}^4$ in the absence of non-trivial boundary conditions, because we can only *measure* differences in energy, but not a constant, ever present (even if infinite) shift in H .

However, once we couple QFT to gravity, the vacuum energy *curves* spacetime and acts as a *cosmological constant*. The fact that the physical, finite value (after renormalization) of it cannot be calculated, but is a free parameter, is part of the cosmological-constant problem.

Now we can continue with constructing the Fock space. To do so, we postulate a vacuum state $|0\rangle$,

$$a_{\vec{p}} |0\rangle = 0 \quad \forall \vec{p}. \quad (119)$$

One-particle states are defined as

$$|\vec{p}\rangle = a_{\vec{p}}^\dagger |0\rangle. \quad (120)$$

Why is this called a one-particle state?

Mini-Exercise 8. Calculate $H |\vec{p}\rangle$ and from your finding, explain why $|\vec{p}\rangle$ is called one particle state.

Solution.

$$\begin{aligned} H |\vec{p}\rangle &= \int \frac{d^3k}{(2\pi)^3} \omega_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}} a_{\vec{p}}^\dagger |0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3} \omega_{\vec{k}} a_{\vec{k}}^\dagger \underbrace{\left[a_{\vec{k}}, a_{\vec{p}}^\dagger \right]}_{(2\pi)^3 \delta^3(\vec{k}-\vec{p})} |0\rangle + 0 \quad (\text{using } a_{\vec{p}} |0\rangle = 0) \\ &= \omega_{\vec{p}} a_{\vec{p}}^\dagger |0\rangle \\ &= \omega_{\vec{p}} |\vec{p}\rangle. \end{aligned}$$

$\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ is the relativistic energy of a single particle of mass m and momentum $\vec{p}$. This justifies calling $|\vec{p}\rangle$ a one-particle state.

Note that this state contains a single particle of fixed momentum, but completely delocalized. We will look at localized states later.

We can continue with N -particle states

$$|\vec{p}_1 \dots \vec{p}_N\rangle = a_{\vec{p}_1}^\dagger \dots a_{\vec{p}_N}^\dagger |0\rangle \quad (121)$$

and find

$$H |\vec{p}_1 \dots \vec{p}_N\rangle = (\omega_{\vec{p}_1} + \dots + \omega_{\vec{p}_N}) |\vec{p}_1 \dots \vec{p}_N\rangle. \quad (122)$$

So far, these states are not normalized yet. We impose

$$||0\rangle|^2 = \langle 0|0\rangle = 1. \quad (123)$$

Then,

$$\left(a_{\vec{p}}^\dagger |0\rangle \right) \cdot \left(a_{\vec{q}}^\dagger |0\rangle \right) = \langle 0| a_{\vec{p}} a_{\vec{q}}^\dagger |0\rangle = (2\pi^3) \delta^3(\vec{p} - \vec{q}), \quad (124)$$

i.e., one-particle states with different momenta are orthogonal. However, we want to change the normalization in order to account for *Lorentz covariance*.

We want

$$\Lambda |p\rangle = |p'\rangle, \quad \text{if } p'^\mu = \Lambda^\mu{}_\nu p^\nu, \quad (125)$$

such that we can demand

$$\langle p'|q'\rangle = \langle p|q\rangle, \quad (126)$$

i.e., a norm that does not depend on the Lorentz frame. Relativistically normalized states, denoted by $|p\rangle$ to distinguish them from $|\vec{p}\rangle$, are

$$|p\rangle = \sqrt{2\omega_{\vec{p}}} a_{\vec{p}}^\dagger |0\rangle. \quad (127)$$

This result will be derived in the exercises; the corresponding derivation will be added here afterwards.

3.3 Casimir effect

A recurring theme throughout this course will be the result that quantum fluctuations have non-trivial physical consequences. Examples that we will consider later include the result that quantum fluctuations can spontaneously break a symmetry that the classical theory has (Coleman-Weinberg potential) and that quantum fluctuations can produce interactions between photons, even though electromagnetic waves are non-interacting in the classical theory (Euler-Heisenberg Lagrangian). A second, related theme is that in the corresponding calculations, divergences show up. Historically, these led (and actually still lead) to confusion, because it is not always straightforward to see that these divergences occur in unphysical, un-measurable quantities and the physical, measurable quantities are finite. The divergences are treated through regularization and renormalization.

In nature, the Casimir effect occurs for the electromagnetic field (that we will quantize in a few weeks). It is a result of the fact that on a conducting plate, the electric field must vanish (because otherwise it induces a current that counteracts the field). Thus, if we place two parallel, conducting plates in a vacuum, the field has to satisfy boundary conditions, namely that it vanishes at the location of both plates:

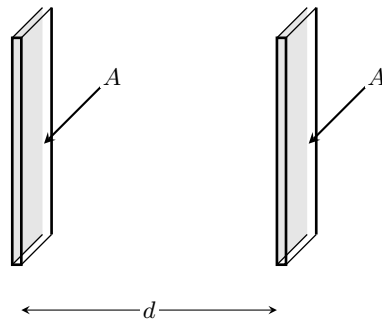


Figure 2: Two parallel plates at distance d , each of area A .

We model this effect with a massless scalar field and simplify the situation to 1+1 dimensions, so that we can forget about the directions parallel to the plates. Then, we impose boundary conditions

$$\phi(0) = 0 = \phi(d). \quad (128)$$

From this, it follows that the momentum is discrete, allowed values are

$$p_x = \frac{n\pi}{d}, \quad (129)$$

so that

$$\omega_{\vec{p}}^2 = \vec{p}^2 \rightarrow \frac{n\pi}{d}, \quad (130)$$

and the integral $\int \frac{dp_x}{2\pi} \rightarrow \frac{1}{d} \sum_{n=1}^{\infty}$.

Thus, the expression for the ground-state energy density is

$$\epsilon = \frac{\langle 0|H|0\rangle}{d} = \frac{1}{d^2} \sum_n \frac{n\pi}{d} \left(\langle 0|a_p^\dagger a_{\bar{p}}|0\rangle + \frac{d}{2} \right) = \frac{1}{2d} \sum_n \frac{n\pi}{d}. \quad (131)$$

In contrast, the energy density outside of the plates is given by a continuous set of Fourier modes. We note that the energy density *between* the plates is therefore *lower* than outside the plates. Thus, we expect a *force* on the plates. This *attractive force* is exerted by the vacuum. It is the Casimir force.

The expression for the ground state energy density is divergent. To calculate the Casimir force, we need to regularize the divergence. We do so by multiplying each mode by $e^{-\alpha \frac{n\pi}{d}}$ and take the limit $\alpha \rightarrow 0$ at the end of the calculation.

Now comes the key physical point about our treatment of (most) divergences in QFT: We need to distinguish divergences in unphysical (i.e., not measurable) quantities from divergences in physical (i.e., measurable quantities). Divergences in unphysical quantities are not necessarily a problem. Divergences in physical quantities are a problem and signal that the theory is not valid for the problem that we have applied it to.

What typically happens in calculations of loop effects (or in other words, calculations that involve quantum fluctuations) in QFT is, that unphysical quantities at some intermediate point of our calculation diverge. Measurable quantities stay finite. In some sense, they just mean that we have not set up our formalism in such a way that is best adapted to the physics.

In the concrete example, we clearly see that the ground-state energy density diverges. However, it is not itself observable. Instead, the Casimir force is observable, so the key question is whether or not the Casimir force diverges, when we remove the regularization.

Then, $F_{\text{Casimir}} = -\frac{\partial d(\Delta\epsilon)}{\partial d}$, where $\Delta\epsilon$ is the difference in energy densities and $d(\Delta\epsilon)$ the difference in energies.

$$\Delta\epsilon = \underbrace{\frac{\pi}{2d^2} \sum_{n=1}^{\infty} n}_{\text{energy density between the plates}} - \lim_{d \rightarrow \infty} \underbrace{\frac{\pi}{2d^2} \sum_{n=1}^{\infty} n}_{\text{energy density outside the plates}}. \quad (132)$$

We write the energy density outside the plates as $\lim_{d \rightarrow \infty}$, so that we can calculate the Casimir force by subtracting two regularized quantities from each other, because the difference of two divergent quantities is ill-defined.

Now we need to regularize the sum $\sum_{n=1}^{\infty} n$, and obtain:

$$F_{\text{Casimir}} = -\frac{\partial}{\partial d} d \lim_{\alpha \rightarrow 0} \left(\frac{\pi}{2d^2} \sum_{n=1}^{\infty} n e^{-\frac{\alpha n \pi}{d}} - \lim_{d \rightarrow \infty} \frac{\pi}{2d^2} \sum_{n=1}^{\infty} n e^{-\frac{\alpha n \pi}{d}} \right). \quad (133)$$

We use

$$\frac{\pi}{2d} \sum_{n=1}^{\infty} n e^{-\frac{\alpha n \pi}{d}} = -\frac{1}{2} \frac{\partial}{\partial \alpha} \sum_{n=1}^{\infty} e^{-\frac{\alpha n \pi}{d}} \quad (134)$$

$$= -\frac{1}{2} \frac{\partial}{\partial \alpha} \frac{1}{1 - e^{-\alpha \pi/d}} \quad (135)$$

$$= \frac{1}{2} \frac{1}{(1 - e^{-\alpha \pi/d})^2} e^{-\frac{\alpha \pi}{d}} \frac{\pi}{d}. \quad (136)$$

Thus,

$$\Rightarrow F_{\text{Casimir}} = -\frac{\partial}{\partial d} d \lim_{\alpha \rightarrow 0} \left(\frac{\pi}{2d^2} \frac{e^{\frac{\alpha\pi}{d}}}{(e^{\alpha\pi/d} - 1)^2} - \lim_{d \rightarrow \infty} \frac{\pi}{2d^2} \frac{e^{\frac{\alpha\pi}{d}}}{(e^{\alpha\pi/d} - 1)^2} \right). \quad (137)$$

Now we take the limit $\alpha \rightarrow 0$, for which we can expand the above expression up to the α^0 term. The series is a Laurent series, i.e., it starts with a negative power of α . This term diverges in the limit $\alpha \rightarrow 0$. This divergence is *not* an observable (measurable) quantity. In the measurable quantity, namely the force, this divergence cancels. As emphasized above, this is a first example of an important point in UV divergences in QFT, namely to carefully distinguish between divergences in physical and unphysical quantities.

In fact, we find that the divergent parts of the energy density between and outside the plates cancel:

$$F_{\text{Casimir}} = -\frac{\partial}{\partial d} d \lim_{\alpha \rightarrow 0} \left(\frac{1}{2\pi\alpha^2} - \frac{\pi}{24d^2} - \lim_{d \rightarrow \infty} \left(\frac{1}{2\pi\alpha^2} - \frac{\pi}{24d^2} \right) \right) \quad (138)$$

$$= -\frac{\partial}{\partial d} \left(-\frac{\pi}{24} \frac{d}{d^2} \right) = -\frac{\pi}{24d^2} \quad (139)$$

Thus, we are finding a finite, attractive force between the plates. The dependence on d has been tested experimentally, confirming our procedure. Note that the key point of properly dealing with the divergences was to focus on an *observable* quantity. In this quantity, two formally divergent expressions cancel, which makes the calculation subtle, but there is nothing problematic about it. Note that our result says that $\sum_{n=1}^{\infty} = -\frac{1}{12}$, at least in the context that we investigated here. In fact, this equality also shows up when doing ζ -function regularization and renormalization. There, the Casimir force is defined from an analytical continuation of the ζ -function, which implies the above result. In this method of dealing with divergent, intermediate results, the regularization and renormalization is done implicitly and it is conceptually less clear what one is doing.

Let us also stress that the final result for the Casimir force is *independent* of the choice of regularization; confirming that we are extracting a prediction for a physical quantity (which must not depend on arbitrarily chosen regularizations).

3.4 Quantum statistics

We know that multi-particle states which are antisymmetric (symmetric) under exchange of any two particles contain fermions (bosons). Which one do we have in our Fock space? Because $[a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0$, we have that $|p_1 p_2\rangle = |p_2 p_1\rangle$ and similarly for states with more particles. In detail, we see the link between the statistics and the choice of commutator (over anticommutator, for which $\{a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger\} = a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger + a_{\vec{q}}^\dagger a_{\vec{p}}^\dagger$ as follows:

$$|p_1 p_2\rangle = a_{\vec{p}_1}^\dagger a_{\vec{p}_2}^\dagger |0\rangle = \left([a_{\vec{p}_1}^\dagger, a_{\vec{p}_2}^\dagger] + a_{\vec{p}_2}^\dagger a_{\vec{p}_1}^\dagger \right) |0\rangle = |p_2 p_1\rangle. \quad (140)$$

This means that the particles are *bosons*. This result is the first example of the spin-statistics theorem, which says that particles with integer spin are bosons and particles with half-integer spin are fermions. Mathematically, the correct quantization for bosons is the one using the commutator of creation and annihilation operators.

If we perform the same calculation with an anticommutator, we obtain $|p_2 p_1\rangle = -|p_1 p_2\rangle$, i.e., the state changes by a sign under the exchange of two particles. This is Fermi-Dirac statistics, and

results in the Pauli exclusion principle, where the state $|pp\rangle = 0$, i.e., no two particles can be in the same state.

In the exercises, we will learn what goes wrong, if we try to quantize using the anticommutator (which would imply fermionic statistics, i.e., the Pauli exclusion principle). When we quantize spin 1/2 particles later in the lecture, we will learn that we must use the anticommutator, and that therefore they satisfy Fermi statistics and the Pauli exclusion principle.

3.5 Interpretation of $\phi(\vec{x})$

Now that we have constructed the Fock space, we can understand how $\phi(\vec{x})$ acts on the vacuum:

$$\phi(\vec{x}) |0\rangle = ? \quad (141)$$

Mini-Exercise 9. Calculate the rhs of this expression. Use

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \tilde{\phi}(\vec{p}) \quad \text{and} \quad \tilde{\phi}(\vec{p}) = \frac{1}{\sqrt{2\omega_{\vec{p}}}} (a_{\vec{p}} + a_{-\vec{p}}^\dagger). \quad (142)$$

Solution.

$$\begin{aligned} \phi(\vec{x}) |0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}} (a_{\vec{p}} + a_{-\vec{p}}^\dagger) |0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}} a_{-\vec{p}}^\dagger |0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} e^{-i\vec{p}\cdot\vec{x}} \underbrace{a_{\vec{p}}^\dagger |0\rangle}_{|\vec{p}\rangle} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-i\vec{p}\cdot\vec{x}} |p\rangle. \end{aligned}$$

This has the interpretation of a superposition of 1-particle states with different momenta. Because we are integrating over all momenta, the result only depends on $\vec{x}$, in other words, $\phi(\vec{x}) |0\rangle$ is a one-particle state at position $\vec{x}$. We will use this information later, when we discuss *causality*.

3.6 Heisenberg picture

So far, we have worked in the Schrödinger picture, where the field operators $\phi(\vec{x})$ and $\Pi(\vec{x})$ have no time-dependence. Instead, the states evolve in time, $|p\rangle = |p(t=0)\rangle$ and $|p(t)\rangle = e^{-iHt} |p(t=0)\rangle$. The relations between field operators and states that we have used so far are relations at $t=0$.

It is clearly not natural in a Poincaré-invariant theory to separate temporal and spatial dependence from each other. Instead, we switch to the *Heisenberg picture*, where operators depend on space *and* time and states are held fixed. The time dependence of any operator $\mathcal{O}$ follows from the requirement

$$\underbrace{\langle\psi|\mathcal{O}(t)|\psi\rangle}_{\text{Heisenberg}} = \underbrace{\langle\psi(t)|\mathcal{O}|\psi(t)\rangle}_{\text{Schrödinger}}. \quad (143)$$

In particular, the field operator becomes

$$\phi(x) = e^{iHt} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right) e^{-iHt} \quad (144)$$

$$= \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}}}_{(*)} \left(a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^\dagger e^{ipx} \right) \quad (145)$$

For the Lorentz invariance of the measure $(*)$, see our discussion of state normalization. Note that we now have 4-vector products in the exponents, i.e., $px = p_\mu x^\mu$, which are also Lorentz invariant.

3.7 Causality

Causality is a key property of a QFT. Causality implies that operators at spacelike distances commute and can therefore be measured simultaneously. It is one of the properties that is not automatically built into Quantum Mechanics and that motivated us to develop a new formalism, the QFT formalism.

In particular, the requirement that operators at spacelike distance should commute holds for $\phi(x)$, i.e., we need $[\phi(x), \phi(y)] = 0$ for $(x - y)^2 < 0$ for causality to hold. Physically, this means that the creation of a particle at x cannot affect the creation of a particle at y , if x and y are spacelike separated.

Note that the equal-time-commutation relations in the Schrödinger picture, $[\phi(\vec{x}), \phi(\vec{y})] = 0$, do not immediately imply causality; we need to check the commutator in the Heisenberg picture to have the full spacetime-dependence of the operator.

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{q}}}} \left([a_{\vec{p}}, a_{\vec{q}}^\dagger] e^{-ipx+iqy} + [a_{\vec{p}}^\dagger, a_{\vec{q}}] e^{ipx-iqy} \right), \quad (146)$$

where we directly set the vanishing commutators among two a 's or two $a^\dagger$'s to zero.

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip(x-y)} - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{ip(x-y)}. \quad (147)$$

This expression does *not* vanish for timelike distances, e.g., taking $(x - y) = (t, 0, 0, 0)$, we get

$$[\phi(\vec{x}, 0), \phi(\vec{x}, t)] \sim e^{-i\omega_{\vec{p}}t} - e^{i\omega_{\vec{p}}t}. \quad (148)$$

This expression vanishes for spacelike separation. This follows, because $[\phi(\vec{x}, t), \phi(\vec{y}, t)]$ vanishes, which we can see from

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_{\vec{p}}} \left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{+i\vec{p}\cdot(\vec{x}-\vec{y})} \right) \quad (149)$$

$$= 0. \quad (150)$$

We have flipped the sign of $\vec{p}$ by changing the integration variable $\vec{p} \rightarrow -\vec{p}$ in the second term. However, $[\phi(x), \phi(y)]$ must be a Lorentz invariant expression, because it is based on the Lorentz invariant integration measure. Thus, it can only depend on $(x - y)^2$ and must therefore vanish for all $(x - y)^2 < 0$, irrespective of whether the two times are equal, since a boost of a spacelike interval can always be used to bring the two points to equal times.

Thus, causality is respected by our theory - which is not surprising, because we're basing it on Lorentz invariance and so the notion that nothing can propagate faster than light is built in.

3.8 A note on quantum entanglement:

Locality in QFT implies that local operators, defined at a single spacetime point, commute at spacelike distances. At a first glance, one might wonder whether locality in QFT is not in contradiction to entanglement in QM, where states can be entangled over spacelike distances, e.g., in EPR-states. However, the fact that in QFT local operators commute is *not* at odds with the existence of entangled states. Consider the following example in QM, an entangled state of two spin 1/2 particles, which can be at large spatial distance

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \quad (151)$$

S_1 , measuring the spin of particle 1, and S_2 , measuring the spin of particle 2, commute nevertheless, and the expectation value of S_2 is not changed by measuring S_1 .

Just like the states in QM can be non-local, the states in QFT are generically non-local; in our discussion of the path-integral formalism we will see that states are represented as functionals of the field configuration and depend on non-local information.

3.9 Propagators and causality

Literature suggestion: Peskin/Schröder.

There is a different question we could ask to probe the causal structure of the theory. We prepare a particle at spacetime point y , by acting with $\phi(y)$ on the vacuum. We then ask that the probability amplitude is to find the particle at $\phi(x)$. This is encoded in the *propagator*

$$D(x-y) = \underbrace{\langle 0 | \phi(x)}_{\text{particle at } x} \underbrace{\phi(y) | 0 \rangle}_{\text{particle at } y} = \langle 0 | \phi(x) \phi(y) | 0 \rangle, \quad (152)$$

which is the probability amplitude for the particle to *propagate* from y to x .

We study its structure here for two reasons: First, at a physical level, we will achieve a crucial insight into the physical reason why antiparticles (particles with the same mass and spin, but opposite charges under internal symmetries, e.g., electric charge) must exist. Second, the techniques of complex analysis that we will use, are useful in many other contexts in QFT and beyond, and the propagator is a useful example to practise them with.

We might first expect that the propagator should vanish for spacelike distances. To evaluate $D(x-y)$, we write

$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \frac{1}{\sqrt{2\omega_{\vec{q}}}} \langle 0 | \left(a_{\vec{q}} e^{-iqx} + a_{\vec{q}}^\dagger e^{iqx} \right) \cdot \left(a_{\vec{p}} e^{-ipy} + a_{\vec{p}}^\dagger e^{ipy} \right) | 0 \rangle \quad (153)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip(x-y)}, \quad \text{using } a_{\vec{p}} | 0 \rangle = 0 = \langle 0 | a_{\vec{q}}^\dagger \quad \text{and } [a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{q}). \quad (154)$$

This expression already looks like it will be non-zero for spacelike distances, but let us evaluate it to see what exactly its form is.

First, we reduce the three-dimensional integral to a one-dimensional integral over the modulus of $\vec{p}$, i.e., $|\vec{p}|$. This works as follows: Because we focus on spacelike distances ($(x - y)^2 < 0$), we can pick $x^0 = y^0$ and name $\vec{x} - \vec{y} = \vec{r}$, so that

$$D(x - y) \Big|_{(x-y)^2 < 0} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{i\vec{p} \cdot \vec{r}} \quad (155)$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty d|\vec{p}| \frac{|\vec{p}|^2}{2\omega_{\vec{p}}} \frac{e^{i|\vec{p}||\vec{r}|} - e^{-i|\vec{p}||\vec{r}|}}{i|\vec{p}||\vec{r}|}. \quad (156)$$

In this step, we rewrote

$$\int d^3 p = \int_0^\infty d|\vec{p}| |\vec{p}|^2 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\varphi \quad \text{and} \quad \vec{p} \cdot \vec{r} = |\vec{p}||\vec{r}| \cos \theta. \quad (157)$$

We used

$$\int_0^\pi d\theta \sin \theta e^{i\alpha \cos \theta} = \frac{2 \sin \alpha}{\alpha} \quad \text{and} \quad 2 \sin \alpha = ie^{-i\alpha} - ie^{i\alpha} \quad \text{for } \alpha = |\vec{p}||\vec{r}|. \quad (158)$$

In the next step, we change variables according to $|\vec{p}| \rightarrow -|\vec{p}|$ in the 2nd term, so that the integral, instead of ranging from 0 to ∞ , ranges from $-\infty$ to $+\infty$.

We thus have that

$$D(x - y) \Big|_{(x-y)^2 < 0} = \frac{-i}{2(2\pi)^2} |\vec{r}| \int_{-\infty}^\infty d|\vec{p}| \frac{|\vec{p}| e^{i|\vec{p}||\vec{r}|}}{\sqrt{|\vec{p}|^2 + m^2}}. \quad (159)$$

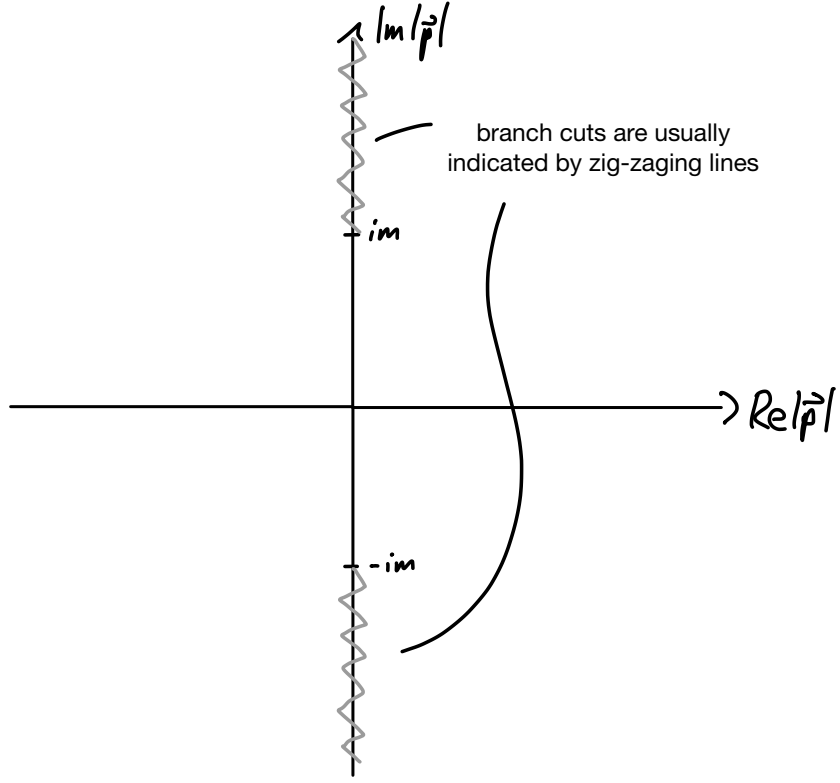
This is actually an integral representation of a Hankel function, so we could just use that result, together with the asymptotic form of the particular Hankel function for large r to obtain $D(x - y) \Big|_{(x-y)^2 < 0} \sim e^{-m r}$. However, to get a better idea why that happens, and to practise techniques of complex analysis, we will arrive at that result differently.

In the complex plane (in this case, the complex- $|\vec{p}|$ -plane), we can use Cauchy's integral theorem. The theorem says that the integral of a function over a closed curve in the complex plane vanishes, if the function is infinitely differentiable and locally identical to its Fourier series (i.e., if it is a holomorphic function). (If the function has poles, then the integral will pick up the corresponding residue.) This allows us to *deform* integration contours, by starting from the integral along the real axis and completing it by some contour through the complex plane, so that together, the real axis and the new part of the contour form a closed curve. By the theorem, the original integral along the real axis is equal to (minus) the integral along that contour.

What we have to watch out for when doing such deformations, are poles (where the function is singular) and branch cuts (where the limit of the function, taken from both sides of the branch cut, is not equal, i.e., the function is multi-valued).

We encounter a simple example of a branch cut for the function $\sqrt{z}$ with $z \in \mathbb{C}$. This example will be relevant for our case. In the complex plane, we can parameterize z through its modulus and a phase

$$\sqrt{z} = \sqrt{r} e^{i\theta/2} = \sqrt{r} \left(\cos \left(\frac{\theta}{2} \right) + i \sin \left(\frac{\theta}{2} \right) \right), \quad \text{with } \theta \in (-\pi, \pi). \quad (160)$$



For $z \in \mathbb{R}$, we have that $\theta = 0$. Positive imaginary z has $\theta = \pi/2$ and negative imaginary z has $\theta = -\pi/2$. Negative real z can be approached coming from positive imaginary parts (with $\theta \rightarrow \pi$) or from negative imaginary parts (with $\theta \rightarrow -\pi$). It turns out that $\sqrt{z}$ is discontinuous across the negative real axis, i.e., it has a branch cut. We can see the discontinuity as follows:

$$\lim_{(r,\theta) \rightarrow (r_0,\pi)} \sqrt{r} \left(\cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right) \right) = \sqrt{r_0} i, \quad (161)$$

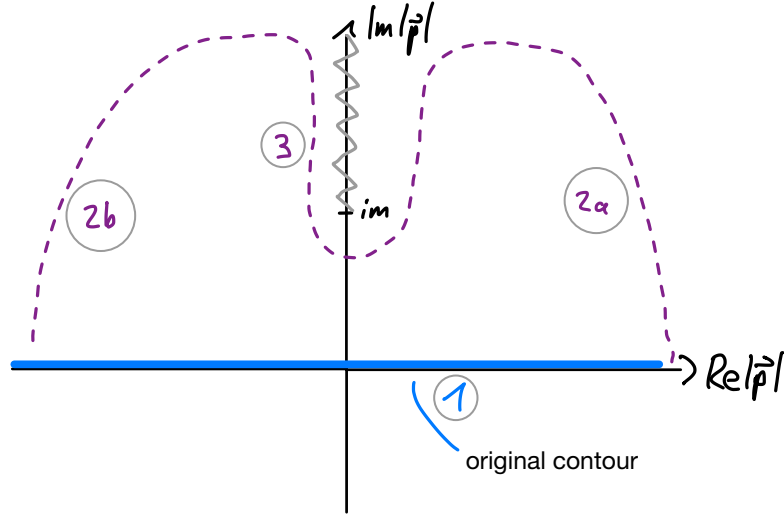
$$\lim_{(r,\theta) \rightarrow (r_0,-\pi)} \sqrt{r} \left(\cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right) \right) = \sqrt{r_0} (-i). \quad (162)$$

In the integral that we are interested, there is a branch cut when $|\vec{p}|^2 + m^2$ becomes negative. This translates into $|\vec{p}|$ being purely imaginary and the imaginary part either being positive and greater than m , or negative and smaller than $-m$.

Therefore, there are branch cuts on the imaginary axis, starting at $\pm im$. This means that the integrand is discontinuous across this line:

$$\lim_{\varepsilon \rightarrow 0} \frac{|\vec{p}| e^{i|\vec{p}|r}}{\sqrt{|\vec{p}|^2 + m^2}} \Big|_{|\vec{p}|=im+\varepsilon} \neq \lim_{\varepsilon \rightarrow 0} \frac{|\vec{p}| e^{i|\vec{p}|r}}{\sqrt{|\vec{p}|^2 + m^2}} \Big|_{|\vec{p}|=im-\varepsilon} \quad (163)$$

The branch cuts start in the singular branch points at $|\vec{p}| = \pm im$. In the presence of a branch cut, we shift the integration contour upwards, so that it wraps around the branch cut, but never crosses it. The difference between the original contour and this contour vanishes because of Cauchy's theorem: Due to Cauchy's residue theorem, the integral along the curve $C = \textcircled{1} + \textcircled{2a} + \textcircled{3} + \textcircled{2b}$



vanishes, because there's no singularity that is enclosed. In addition, the integrals along (2a) and (2b) (which are meant to lie at infinity) vanish. Thus, the integral along (1) is equal to the integral along (3) (followed from right to left).

Because we saw previously that $\sqrt{z}$ differs by an overall sign across the branch cut, the integral along the right and the left part of the branch cut add up and we obtain

$$\begin{aligned}
D(x-y) \Big|_{(x-y)^2 < 0} &= \frac{-i}{2(2\pi)^2} |\vec{r}| \lim_{\epsilon \rightarrow 0} \left(\int_{i\infty+\epsilon}^{im+\epsilon} d|\vec{p}| \frac{|\vec{p}| e^{i|\vec{p}||\vec{r}|}}{\sqrt{|\vec{p}|^2 + m^2}} + \int_{im-\epsilon}^{i\infty-\epsilon} d|\vec{p}| \frac{|\vec{p}| e^{i|\vec{p}||\vec{r}|}}{\sqrt{|\vec{p}|^2 + m^2}} \right) \\
&= \frac{-i}{(2\pi)^2} |\vec{r}| \int_{im}^{i\infty} d|\vec{p}| \frac{|\vec{p}| e^{i|\vec{p}||\vec{r}|}}{\sqrt{|\vec{p}|^2 + m^2}} \\
&= \frac{1}{(2\pi)^2} \frac{1}{|\vec{r}|} \int_m^\infty d\rho \rho \frac{e^{-\rho|\vec{r}|}}{\sqrt{\rho^2 - m^2}}, \tag{164}
\end{aligned}$$

where in the last step we defined $\rho = -i|\vec{p}|$.

We are only interested in the behavior at $rm \gg 1$, i.e., distances large compared to the scale set by the mass, in which case the integrand is suppressed for all values of ρ , except for the point $\rho = m$. Thus we obtain

$$D(x-y) \Big|_{(x-y)^2 < 0} \stackrel{|\vec{r}| m \gg 1}{\approx} e^{-m|\vec{r}|}. \tag{165}$$

Thus, there is a nonzero probability amplitude for a particle to propagate outside the lightcone. This seems worrisome. Does this mean that our theory violates causality? The answer is no, because this does not affect *measurements*, i.e., it is a property of the theory we can never test in an experiment (or use for faster-than-light signalling). In fact,

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) = 0 \quad \text{for } (x-y)^2 < 0. \tag{166}$$

Thus, the quantum amplitude for the particle to propagate over a spacelike distance from y to x *interferes destructively* with the quantum amplitude for a particle to propagate from x to y . Such a destructive interference makes sense, because, if x and y are spacelike to each other, the temporal

order of the two is not fixed and can be changed. Therefore, it is equally viable to consider propagation from x to y as it is from y to x and thus both processes occur. Their quantum amplitudes cancel when we consider a measurement.

Note that this would be different, if x and y were timelike to each other, because then only one direction of travel makes sense; the other one would be against the direction of time.

In a complex scalar theory, it gets even more interesting, because we can consider the commutator

$$[\phi(x), \phi^\dagger(y)] = \underbrace{\langle 0 | \phi(x) \phi^\dagger(y) | 0 \rangle}_{\text{describes particle propagating from } y \text{ to } x} - \underbrace{\langle 0 | \phi^\dagger(y) \phi(x) | 0 \rangle}_{\text{describes antiparticle propagating from } x \text{ to } y}. \quad (167)$$

This is a profound result: it tells us that antiparticles *must* exist for QFT to be causal. It is *not* a choice to have or not have antiparticles. They are a *necessity* for a causal theory. (In the case of a real scalar field we did not see that clearly, because an uncharged scalar, described by a real scalar field, is its own antiparticle.)

4 Path-integral quantization for the scalar field

(See Gelis, Ryder, Franklin, Srednicki)

This approach to quantum physics places the superposition principle - that more than one classical realization of a system is realized at the same time - center stage. For a single particle, it means that if it is in a momentum eigenstates, it is in a superposition of position eigenstates, e.g., completely delocalized. For a field, it means that the relevant entity is *not* a single configuration of the scalar field (which we can associate to particles at some positions). Rather, all field configurations which are compatible with boundary of initial conditions are realized at the same time and interfere destructively or constructively. In terms of particles, this means that we never have a constant number of particles, but that in addition to the real particles, there are *virtual* particles in our theory, which are only there for short amounts of time, before they disappear again.

Mathematically, instead of working with operators on a Hilbert space, we work with functionals, i.e., maps from the space of functions (field configurations) to the real (complex) numbers. We will introduce the necessary mathematical concepts as we go along and partially in the exercises.

Conceptually, besides providing a different (complementary) intuition about QFT than the canonical formalism, the path-integral formalism is also manifestly Lorentz invariant.

In Quantum Mechanics and in Quantum Field Theory, the canonical formalism and the path-integral formalism are equivalent. In quantum gravity, the situation is unclear. There are approaches to quantum gravity, in which spacetime as a whole is quantized in such a way that a Hamiltonian cannot be written down, just an action and a path integral (e.g., causal set theory).

Our plan for the next few lectures is to:

- derive path integral for QM
- generalize path integral to QFT

- introduce interactions
- first application: introduce interactions and understand the effect of quantum fluctuations: symmetry breaking in the one-loop effective potential.

4.1 Path-integral for Quantum Mechanics

We will now derive an expression for the probability amplitude for a particle to propagate from position q_i to position q_f that is a superposition of all paths, each one weighted with a complex amplitude that results in (constructive or destructive) interference between paths.

Consider QM for a single classical degree of freedom, described by the Hamiltonian

$$H = \frac{P^2}{2m} + V(Q), \quad (168)$$

with momentum operator P and position operator Q , and commutator $[Q, P] = i$. The probability for the particle to start at the initial position q_i and end at the final position q_f after time $t_f - t_i$ is given by:

$$\left| \langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle \right|^2. \quad (169)$$

We will derive the path-integral expression for the amplitude. The intuition underlying the path integral can be obtained by starting from the double slit, in which this transition has two major contributions, one from each path, see left panel in Fig. 3. Then we imagine generalizing the 2 slits to n slits and the one barrier to m barriers, see right panel in Fig. 3.

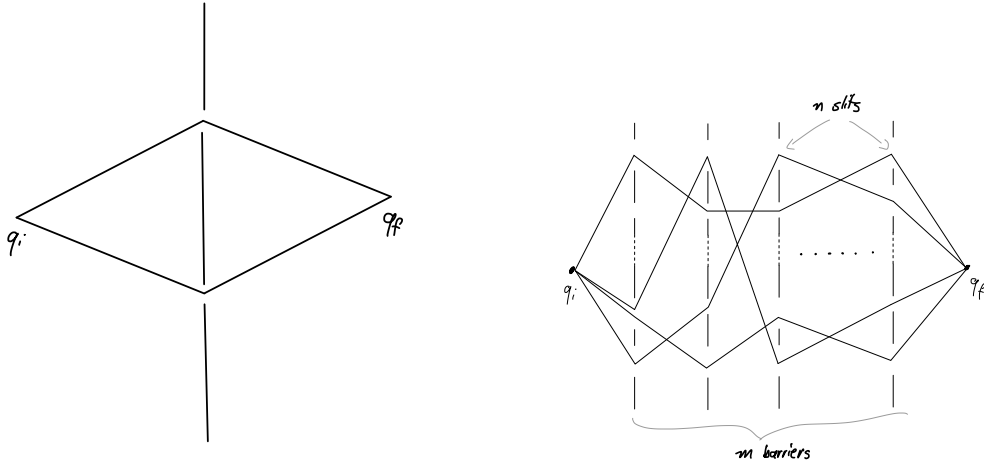


Figure 3: Left panel: we show the two possible paths in the double-slit experiment. Right panel: we show the generalization to m barriers with n slits in each and a subset of all possible paths in this setup.

As we take $n \rightarrow \infty$ and $m \rightarrow \infty$, we obtain all possible paths between q_i and q_f that a freely propagating particle can take, see Fig. 4.

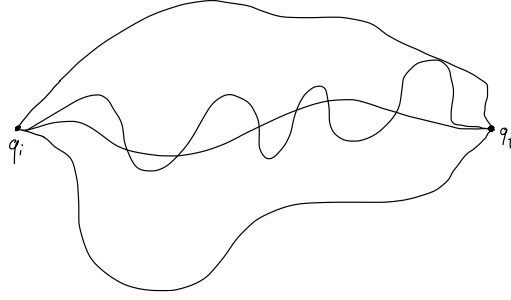


Figure 4: The generalization contains all possible paths that we can imagine, some of which are shown in this sketch to illustrate the concept.

This is what the propagating particle (in a potential $V(Q)$) does. Just like the two paths interfere in the double-slit setup, all paths for the freely propagating particle interfere. As it turns out, paths far away from the classical path interfere destructively, so that the transition probability amplitude is dominated by the classical path and fluctuations around it.

Before we derive the expression for the path integral and the expression for the quantum amplitude associated to each path, we can already develop some intuition for what the result should be. We take the following steps in our argument that motivates the result:

- 1) We are looking for a quantum amplitude, i.e., a quantity that can in general be complex and whose absolute value can never exceed 1, so that it has a probability interpretation. This suggests that we can write the quantum amplitude as $\exp[iA]$, with some A that we need to determine and that is real and depends on the path that we are considering, i.e., $A = A[\text{path}]$.
- 2) We consider how constructive and destructive interference between different paths can be encoded in this quantity: if we compare the quantum amplitudes $\exp[iA[\text{path}]]$ for two different paths, and they differ exactly by their sign, then they cancel in the final expression. In contrast, if the amplitudes are close to each other in their value in the complex plane, then they contribute to the final expression.
- 3) We consider the classical path for the freely propagating particle. We know from Quantum Mechanics lectures that the expectation value of the position of the particle should follow the classical path. Therefore, we want that $\exp[iA[\text{path}]]$ varies slowly across different paths that are close to the classical one. This already gives us a hint, what $A[\text{path}]$ could be, because we know that the *action* has an extremum for the classical path and therefore varies slowly for paths close to it.
- 4) We cross-check our expectation that $\exp[iS[\text{path}]]$, with S the action, could be a good candidate for the quantum amplitude for each path with the expectation that A should be a Lorentz-invariant expression, which S indeed is.

This line of reasoning leads us to a *well-motivated expectation* that the path integral may contain a factor e^{iS} for each path that will be included.

Of course this is absolutely *not* a derivation; it's just a way to obtain some intuition for what we might expect. We will now do a proper derivation of the path integral.

For the derivation, our ultimate goal is to start from the transition amplitude, written in terms of operators acting on states, and convert it into a (function) integral over all paths, where each path is weighted by a quantum amplitude that is a *complex number, not an operator*. Thus, the steps of our derivation will be aimed at getting rid of operators and exchanging them for eigenvalues. We subdivide the time interval $[t_i, t_f]$ into N small time intervals, so that we can keep track of what the particle does in the small intervals. We introduce

$$\Delta = \frac{t_f - t_i}{N}, \quad t_n = t_i + n\Delta, \quad (170)$$

so $t_0 = t_i$, $t_N = t_f$, so that we can write the transition amplitude in terms of the many intermediate positions.

Now we can factorize

$$e^{-iH(t_f - t_i)} = e^{-iH(t_N - t_{N-1})} e^{-iH(t_{N-1} - t_{N-2})} \dots e^{-iH(t_1 - t_0)}, \quad (171)$$

which is possible because H at successive times commute. Between these successive factors on the right-hand-side, we can insert the identity operator as a sum over position eigenstates, in order to express the transition amplitude in terms of the positions at successive times:

$$\mathbb{1} = \int_{-\infty}^{\infty} dq |q\rangle\langle q|. \quad (172)$$

Because we will need the identity operator multiple times, and it would lead to confusion if each of the dummy integration variables was called q , we will instead introduce dummy variables q_j with $j = 1, 2, \dots$ for the identity operator inserted at successive times $t_1, t_2, \dots$. This will eventually allow us to convert the $V(Q)$ in H into $V(q_n)$, the position at time t_n .

$$\langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle = \int \prod_{j=1}^{N-1} dq_j \langle q_f | e^{-iH\Delta} | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\Delta} | q_{N-2} \rangle \dots \langle q_1 | e^{-iH\Delta} | q_i \rangle. \quad (173)$$

Because the time interval consists of N parts, there are $N - 1$ factors of $\mathbb{1}$ to insert inbetween successive factors of $e^{-iH\Delta}$. Now, there is a slight complication, namely that the two terms in H , $\frac{P^2}{2m}$ and $V(Q)$, do not commute, because $[Q, P] = i$. This makes it difficult to split $e^{iH\Delta}$ into $e^{i\frac{P^2}{2m}\Delta}$ and $e^{iV(Q)\Delta}$ and use $e^{iV(Q)} |q_n\rangle = e^{iV(q_n)} |q_n\rangle$.

However, we can use a trick: We can use the Baker-Campbell-Hausdorff-formula:

$$e^{\Delta(A+B)} = e^{\Delta A} e^{\Delta B} e^{-\frac{\Delta^2}{2}[A,B] + \mathcal{O}(\Delta^3)}. \quad (174)$$

For $\Delta \rightarrow 0$ (i.e., $N \rightarrow \infty$), the Δ^2 - and all higher-order factors are negligible. Then, in the limit $\Delta \rightarrow 0$, we can use

$$\langle q_{i+1} | e^{-i\Delta H} | q_i \rangle = \langle q_{i+1} | e^{-i\Delta \frac{P^2}{2m}} e^{-i\Delta V(Q)} | q_i \rangle + \mathcal{O}(\Delta^2) \quad (175)$$

$$= \langle q_{i+1} | e^{-i\Delta \frac{P^2}{2m}} e^{-i\Delta V(q_i)} | q_i \rangle + \mathcal{O}(\Delta^2). \quad (176)$$

This contains the eigenvalue q_i of the position operator, but it still contains the momentum operator P itself. To convert it into its eigenvalue, we insert the identity operator written in terms of

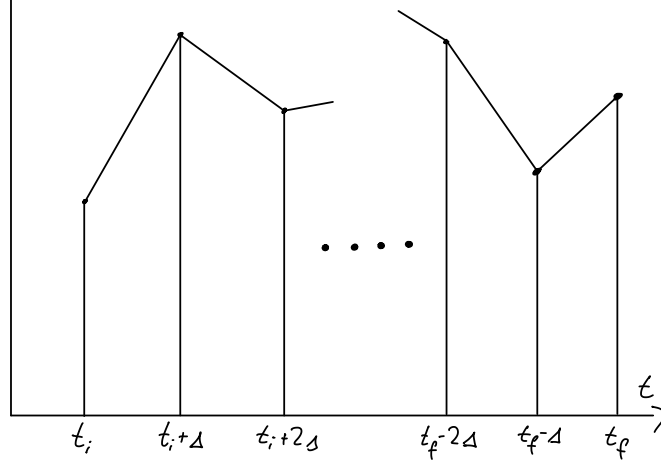


Figure 5: We illustrate the position of the particle at consecutive moments in time, as it enters the expression (181). There is no actual “path”; the connecting lines between the points are just to guide the eye. The “paths” are *not* continuously differentiable.

momentum eigenstates

$$\int \frac{dp}{2\pi} |p\rangle\langle p| = \mathbb{1}. \quad (177)$$

Just like for the position operator, we will use a dummy integration variable labelled by the time at which we insert the identity into the whole expression. Thus we obtain

$$\langle q_{i+1} | e^{-i\Delta H} | q_i \rangle = \int \frac{dp}{2\pi} \langle q_{i+1} | e^{-i\Delta \frac{p_i}{2m}} | p_i \rangle e^{-i\Delta V(q_i)} \langle p_i | q_i \rangle + \mathcal{O}(\Delta^2). \quad (178)$$

We already exchanged the operators in our expression for their eigenvalues, thus we can also pull the factor $e^{-i\Delta \frac{p_i}{2m}}$ out of the matrix element. However, we still have states, which we want to exchange for (complex) numbers next. Thus, we use that

$$\langle q | p \rangle = e^{ipq}, \quad (179)$$

and arrive at

$$\langle q_{i+1} | e^{-i\Delta H} | q_i \rangle = \int \frac{dp_i}{2\pi} e^{-i\Delta H(p_i, q_i)} e^{ip_i(q_{i+1} - q_i)} + \mathcal{O}(\Delta^2). \quad (180)$$

Note that $H(p_i, q_i)$ is now a *number*, no longer an operator and the right-hand side in general no longer contains operators (just their eigenvalues), nor states.

It remains to interpret $e^{-i\Delta H(p_i, q_i)} e^{ip_i(q_{i+1} - q_i)}$. To do so, we next note that $\frac{q_{i+1} - q_i}{\Delta} = \dot{q}_i$ is the discretization of the time-derivative of q_i . Overall, we thus have

$$\langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle = \lim_{N \rightarrow \infty} \left(\int \prod_{j=1}^{N-1} dq_j \right) \left(\prod_{j=1}^N \frac{dp_j}{2\pi} \right) e^{-i\Delta H(p_j, q_j)} e^{ip_j \dot{q}_j}. \quad (181)$$

Let us illustrate this expression, see Fig. 5. Note that there are $N - 1$ integrations over q_j , because we inserted a $\mathbb{1}$ expressed in position eigenstates inbetween each of the factors of $e^{i\Delta H}$. This gives rise to N matrix elements of the form $\langle q_{i+1} | e^{-i\Delta H} | q_i \rangle$ and we insert a $\mathbb{1}$ expressed in terms of momentum eigenstates inside each such matrix elements, so that we end up with N integrations over p_j .

Taking the $N \rightarrow \infty$ limit, we see that $e^{-i\Delta H(p_i, q_i)}$ will become $e^{-i \int_{t_i}^{t_f} dt H(p(t), q(t))}$ (and similarly for the other factor in the exponential). We thus obtain

$$\langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle = \mathcal{N} \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}p(t) \mathcal{D}q(t) e^{i \int_{t_i}^{t_f} dt (p(t)\dot{q}(t) - H(p, q))}, \quad (182)$$

where we introduced a normalization $\mathcal{N}$. This is the path-integral representation of the transition amplitude in its phase-space form. The measures $\mathcal{D}p(t)$ and $\mathcal{D}q(t)$ indicate that we are *not* integrating over *numbers* dp, dq , but over *functions* $q(t)$ and $p(t)$. The path-integral is therefore a *functional* integral.

Comments:

- There are no initial and final conditions on the momentum, because the position is sharp at beginning and end, and so the momentum must be totally unconstrained.
- The right-hand-side contains ordinary commuting numbers, while the left-hand-side contains operators.
- We *derived* the path integral from the canonical formulation of QM. However, we may also “forget” about this origin and view the path integral as the *definition* of the quantum theory.
- The physical intuition behind the path integral is that all possible phase-space configurations $\{p(t), q(t)\}$ are realized simultaneously and each is weighted by a *phase* factor $e^{i \int dt (p\dot{q} - H)}$, which encodes quantum mechanical interference.
- The functional measure $\mathcal{D}q(t)\mathcal{D}p(t)$ in general lacks a solid mathematical foundation. Nevertheless, the formalism allows us to make progress in QFT, where the canonical formalism would prove extremely challenging and cumbersome. Wherever both formalisms can be used in practical calculations, they produce results that are in agreement.

In QFT, the form of the path integral that is used most is not a phase-space path integral, but one, where the momentum-integration has already been performed.

This can be done in theories in which the momentum only occurs quadratically, as we have assumed. In this case, we can perform the integral over $p(t)$, because it is a *Gaussian functional integral*.

We generalize

$$\int_{-\infty}^{\infty} dx e^{\pm \frac{ix^2}{2\sigma}} = \sqrt{2\pi\sigma} e^{\pm i\frac{\pi}{4}}. \quad (183)$$

(Note that this looks as if we took $\int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma}} = \sqrt{2\pi\sigma}$ and simply made it complex. This is not the case. The correct derivation relies on the integral along the real axis, but then uses Cauchy’s residue theorem.)

To use this, we go back to the path integral *before* taking the $N \rightarrow \infty$ limit. We treat each p_i as an independent variable and perform the Gaussian integral to obtain

$$\int \mathcal{D}p e^{i\Delta(p\dot{q} - \frac{p^2}{2m})} = \underbrace{e^{i\frac{\pi}{4}} \sqrt{\frac{2\pi m}{\Delta}}}_{\text{this prefactor is independent of } q, \dot{q}} e^{i\frac{\Delta m \dot{q}^2}{2}}. \quad (184)$$

We absorb the prefactor into the (undetermined) normalization constant $\mathcal{N}$. We will soon see that $\mathcal{N}$ drops out of computing *observables*. Therefore, it does not matter that $\mathcal{N}$ contains a factor that diverges for $\Delta \rightarrow 0$. We thus arrive at the path integral in the form

$$\langle q_f | e^{-iH(t_f-t_i)} | q_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q(t) e^{i \int_{t_i}^{t_f} dt \mathcal{L}(q(t))}, \quad (185)$$

because $L = \frac{m\dot{q}^2}{2} - V(q)$. Therefore,

$$\langle q_f | e^{-iH(t_f-t_i)} | q_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q(t) e^{iS[q(t)]}. \quad (186)$$

This is also known as the Feynman-Kac-formula.

4.2 Classical action, least-action principle

We have worked with $\hbar = 1$, so let us briefly reinstate $\hbar$. We know that $[S] = [\hbar]$ and we know that the argument of the exp cannot contain units. Thus, without redoing the calculation with $\hbar$ back in place, we know that the result has to be

$$\langle q_f | e^{-i\frac{H}{\hbar}(t_f-t_i)} | q_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q e^{\frac{i}{\hbar}S[q]}. \quad (187)$$

The rhs is a wildly oscillatory integral, with the following properties:

- 1) $e^{\frac{i}{\hbar}S}$ is a pure *phase factor*, i.e., $\left| e^{\frac{i}{\hbar}S} \right| = 1$ for all $q(t)$.
- 2) If $S[q(t)]$ changes slowly over neighboring paths, these contribute with a similar phase to the overall integral, i.e., they interfere *constructively*.
- 3) If $S[q(t)]$ changes rapidly across neighboring paths, we find that the contributions *cancel* each other, because they, roughly speaking, contribute with opposite sign.

Mini-Exercise 10. Based on properties 2) and 3), argue why you would expect a main contribution to the path integral from the classical paths, i.e., those paths that satisfy the classical equations of motion, and paths close to them.

Solution. Classical paths satisfy $\delta S = 0$, i.e., the action varies slowly.

4.3 Time-ordered products and generating functional

Before we make the transition to QFT, we need to develop a few more tools. With the QFT-application in mind, we will in particular develop those tools useful in QFT, which may not be the most important quantities in QM. Thus, why we are focusing on the specific quantities that we are will become clearer later in the course. For now we are just developing a toolbox, and we will come back to asking physical questions soon. When we will do so, we will have the tools available to answer them.

We are not only interested in transition amplitudes, but also in other quantities, e.g., expectation values. We will now see that these also have a path-integral representation. Consider the matrix element

$$\langle q_f | e^{-iH(t_f-t_I)} Q e^{-iH(t_I-t_i)} | q_i \rangle, \quad (188)$$

which measures the amplitude for the transition between q_i and q_f with Q acting at the intermediate time t_I , with $t_i < t_I < t_f$. To evaluate this, we write

$$Q = \int dq dq' |q\rangle \underbrace{\langle q| Q |q'\rangle}_{\delta(q-q')} \langle q'| \quad (189)$$

$$= \int dq q |q\rangle \langle q|. \quad (190)$$

By repeating the remainder of our previous path-integral derivation, but with this insertion, we arrive at

$$\langle q_f | e^{-iH(t_f-t_I)} Q e^{-iH(t_I-t_i)} | q_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q q(t_I) e^{iS[q]}. \quad (191)$$

Similarly, we can consider several Q 's inserted at different times. In QM, these are not necessarily the quantities we are most interested in. However, as we will see later in the course, when we discuss the scattering of particles, the analogous expressions in QFT are central to the theory. Thus, we check that they can be written in terms of the path integral in QM. If we have $Q(t_1)Q(t_2)$, with $Q(t) = e^{iHt} Q e^{-iHt}$, we see that the order of the Q 's matters, because Q does not commute with the P^2 -term inside H . Thus we consider the time-ordered products

$$TQ(t_1)Q(t_2) = \begin{cases} Q(t_1)Q(t_2), & \text{if } t_1 \geq t_2 \\ Q(t_2)Q(t_1), & \text{if } t_2 > t_1 \end{cases} \quad (192)$$

(with $Q(t) = e^{iHt} Q e^{-iHt}$ and $|q, t\rangle = e^{iHt} |q\rangle$.)

We have, by the same procedure of inserting $Q(t_2/t_1)$ in the derivation,

$$\langle q_f, t_f | T(Q(t_1)Q(t_2)) | q_i, t_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q q(t_1)q(t_2) e^{iS[q]}. \quad (193)$$

Note that the right-hand-side contains *commuting numbers* $q(t_1)$, $q(t_2)$, thus the time-ordering on the left is crucial for the equality, because otherwise we would have to account for extra commutators that have no counterpart on the right-hand-side. The generalization to n factors is direct

$$\langle q_f, t_f | T(Q(t_1) \dots Q(t_n)) | q_i, t_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q q(t_1) \dots q(t_n) e^{iS[q]}. \quad (194)$$

In QM, these are not necessarily the matrix element that are most interesting for us. In QFT, however, the generalization $q(t) \rightarrow \text{field}(t)$ with such time-ordered correlators will be absolutely crucial. Therefore, it will be very useful to us to “package” all such matrix elements with n factors into a *generating functional*

$$Z_{fi}[\underset{\substack{\uparrow \\ \text{source}}}{j(t)}] = \langle q_f, t_f | T e^{i \int_{t_i}^{t_f} dt j(t) Q(t)} | q_i, t_i \rangle, \quad (195)$$

from which

$$\langle q_f, t_f | T(Q(t_1) \dots Q(t_n)) | q_i, t_i \rangle = \frac{\delta^n Z_{fi}[j]}{i^n \delta j(t_1) \dots \delta j(t_n)} \Big|_{j=0}. \quad (196)$$

The path-integral representation of the generating functional is

$$Z_{fi}[j(t)] = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q \, e^{iS + i \int_{t_i}^{t_f} dt j(t) q(t)}. \quad (197)$$

4.4 Functional differentiation

Above we have introduced *functional* derivatives, denoted by $\frac{\delta}{\delta j(t)}$, which are derivatives with respect to a *function*, not with respect to a c-number quantity. Instead of $\frac{dx}{dx} = 1$, it therefore holds that

$$\frac{\delta f(x)}{\delta f(x')} = \delta(x - x'), \quad (198)$$

with $\delta(x - x')$ being the “1” in the space of functions.

More formally, just like the limit of a finite difference defines a standard derivative, we can define a functional derivative of the functional $F[\phi]$ by:

$$\int dx \frac{\delta F[\phi]}{\delta \phi(x)} \epsilon(x) + \mathcal{O}(\epsilon^2) = F[f + \epsilon] - F[f], \quad (199)$$

where $\epsilon(x)$ is understood as a small change of $\phi(x)$, i.e., it should be small everywhere (and possibly with compact support).

We denote functionals by angular brackets around their arguments (which are functions). Note that a functional maps a *function* to a *number*. The action $S[\phi]$ is a good example: It takes a function $\phi(x)$, and, by integrating its Lagrange density $\mathcal{L}(\phi)$ (which is itself a function of x through its dependence of ϕ and its derivative at each individual point), maps the whole configuration ϕ to a single number.

Based on the definition Eq. (199), we can derive that

$$\frac{\delta}{\delta \phi(y)} \phi(x) = \delta^d(x - y), \quad (200)$$

as follows (we’ll write the derivation for $d = 1$): We consider the particular functional

$$F_\delta[f] = \int_{-\infty}^{\infty} dx \, \delta(x - y) f(x). \quad (201)$$

From the definition Eq. (199), it follows that

$$\int dx \frac{\delta F_\delta[f]}{\delta f(x)} \epsilon(x) = \int_{-\infty}^{\infty} dx \, \delta(x - y) (f(x) + \epsilon(x)) - \int_{-\infty}^{\infty} dx \, \delta(x - y) f(x) = \epsilon(y). \quad (202)$$

For the left-hand side to be equal to the right-hand-side, it must hold that

$$\frac{\delta F_\delta[f]}{\delta f(x)} = \delta(x - y). \quad (203)$$

Thus we have that

$$\delta(x-y) = \frac{\delta F_\delta[f]}{\delta f(x)} = \frac{\delta}{\delta f(x)} \int_{-\infty}^{\infty} dx \delta(x-y) f(x) = \frac{\delta}{\delta f(x)} f(y). \quad (204)$$

For functional derivatives, there is a product rule

$$\frac{\delta}{\delta \phi(x)} (F[\phi] G[\phi]) = \frac{\delta F[\phi]}{\delta \phi(x)} G[\phi] + F[\phi] \frac{\delta G[\phi]}{\delta \phi(x)}, \quad (205)$$

and a chain rule

$$\frac{\delta}{\delta \phi(x)} F[G[\phi]] = \int dy \left. \frac{\delta F[G]}{\delta G(y)} \right|_{G=G[\phi]} \frac{\delta G[\phi(y)]}{\delta \phi(x)}. \quad (206)$$

Mini-Exercise 11. Check that

$$\left. -i \frac{\delta Z_{fi}[j(t)]}{\delta j(t_1)} \right|_{j=0} = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q e^{iS[q]} q(t_1). \quad (207)$$

Solution.

$$\left. -i \frac{\delta Z_{fi}[j(t)]}{\delta j(t_1)} \right|_{j=0} = \left. -i \frac{\delta}{\delta j(t_1)} \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q e^{iS+i \int dt j(t) q(t)} \right|_{j=0} \quad (208)$$

$$= \left. -i \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q i \left(\int dt \delta(t-t_1) q(t) \right) e^{iS+i \int dt j(t) q(t)} \right|_{j=0} \quad (209)$$

$$= \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q q(t_1) e^{iS} \quad (210)$$

4.5 Projection onto the ground state at asymptotic times

Suggested reading for this section: Sec. 5.5 in Ryder; QFT I+II lecture notes from Heidelberg U. by Timo Weigand...

In QM, we typically care about transition states between states that are not the ground state. In QFT, for many problems, we are actually interested in a vacuum-to-vacuum transition amplitude, or the correlators evaluated in the ground state. The reason is twofold:

First, as we already had a glimpse of when we discussed the Casimir effect, already the vacuum (or ground state) is quite non-trivial in a QFT.

Second, we often care about a description of particle scattering events (e.g. at the LHC, or in a cosmic-ray shower, or in the IceCube neutrino detector...), where the particle(s) of interest are *created* (e.g., by collision), they interact and then they are destroyed (e.g. by detection).

The act of creation may be represented by a *source* and that of destruction (which is also, in some sense, a source). The boundary conditions of the problem may then be represented as in the following figure, where the vacuum at $t = -\infty$ evolves into the vacuum at $t \rightarrow +\infty$, via the creation, interaction and destruction of particles, through the agency of a source. We are thus interested in

$$\langle 0, \infty | 0, -\infty \rangle. \quad (211)$$

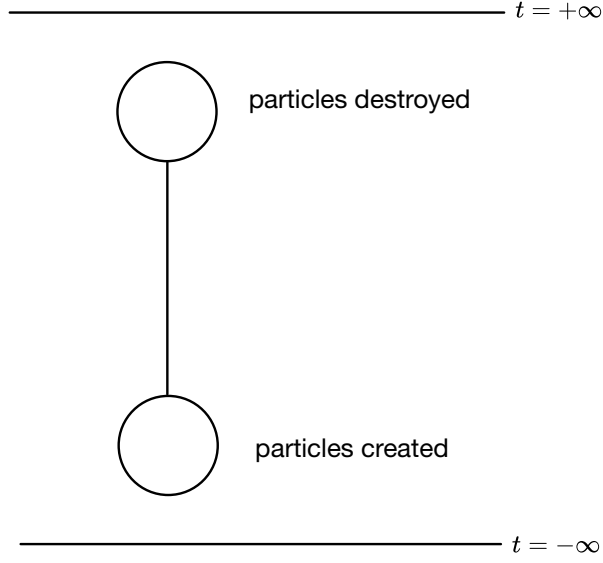


Figure 6: We sketch the idea behind the source acting over a finite amount of time.

How do we obtain it from

$$\langle Q', T' | Q, T \rangle = \int_{\substack{q(T')=Q' \\ q(T)=Q}} \mathcal{D}q e^{i \int_T^{T'} dt (L + Jq)} \quad ? \quad (212)$$

We will consider this still in QM and then generalize to QFT.

We assume $J = J(t)$ and $J = 0$ for $t < t''$ and $t > t'$, with $T < t''$ and $t' < T'$, i.e., the source is switched on after the asymptotic time T and switched off before the asymptotic time T' .

To distinguish whether the source is present or not, we write $\langle q_1, t_1 | q_2, t_2 \rangle^J$ if J is nonzero for (part of) the time inbetween t_1 and t_2 , $t_2 < t < t_1$ and otherwise, we leave out the J . Thus we consider

$$\langle Q', T' | Q, T \rangle = \int dq' dq \langle Q', T' | q', t' \rangle \langle q', t' | q, t \rangle^J \langle q, t | Q, T \rangle. \quad (213)$$

We can write

$$\langle Q', T' | q', t' \rangle = \langle Q' | e^{-iHT'} e^{iHt'} | q' \rangle, \quad (214)$$

where we used $|q, t\rangle = e^{-iHt} |q\rangle$.

We ultimately want to rewrite this into an expression containing the *ground state* and so it makes sense to use a complete set of *energy* eigenstates next. We write this as

$$|q', t'\rangle = \sum_n e^{iHt'} |n\rangle \langle n | q' \rangle = \underbrace{|0\rangle \langle 0 | q' \rangle}_{\text{assuming } E_0=0} + \sum_{|n\rangle \neq |0\rangle} e^{iE_n t'} |n\rangle \langle n | q' \rangle. \quad (215)$$

For simplicity, we symbolize the energy eigenstates by the label “ n ” and write a sum over them. It doesn’t matter for our derivation if they are actually continuous.

This leads to

$$\langle Q', T' | q', t' \rangle = \sum_n \langle n | e^{iE_n(t'-T')} | n \rangle \langle Q | n \rangle \langle q' | n \rangle \quad (216)$$

$$= \langle 0 | 0 \rangle \langle Q' | 0 \rangle \langle 0 | q' \rangle + \sum_{|n\rangle \neq |0\rangle} \langle n | e^{iE_n(t'-T')} | n \rangle \langle Q' | n \rangle \langle n | q' \rangle. \quad (217)$$

To isolate the contribution from the vacuum state, which is the one that we are interested in, we change $T' \rightarrow \infty(1 - i\varepsilon)$ and $T'' \rightarrow -\infty(1 - i\varepsilon)$, with $\varepsilon > 0$. Then,

$$e^{iE_n(t'-T')} \rightarrow e^{iE_n(t'-T'(1-i\varepsilon))} = e^{iE_n(t'-T') - \varepsilon T' E_n}. \quad (218)$$

The term $\sim \varepsilon$ is an *exponential suppression factor*. In the limit $T' \rightarrow \infty$, it results in a suppression of all states that are not the ground state. Thus, we obtain

$$\lim_{\substack{T' \rightarrow \infty(1-i\varepsilon) \\ T \rightarrow -\infty(1-i\varepsilon)}} \langle Q', T' | Q, T \rangle \sim \langle 0, \infty | 0, -\infty \rangle, \quad (219)$$

or, in terms of the path integral:

$$\langle 0, \infty | 0, -\infty \rangle = \mathcal{N} \int \mathcal{D}q e^{i \int_{-\infty}^{\infty} (L + Jq + \frac{1}{2} i\varepsilon q^2)}. \quad (220)$$

Note that we have dropped proportionality factors, that we just absorb in the overall normalization of the path integral. They drop out of physical observables.

Similarly, for vacuum expectation values of operators, we have a similar path integral. It doesn't have any boundary conditions; it is simply the path integral over *all* functions.

4.6 Path integral in quantum field theory

The path-integral formalism can be generalized from QM to QFT. It provides a framework that in many cases is easier to deal with (e.g., when formulating the theory of the strong interactions, QCD). Also, it provides a conceptually different way of thinking about QFT, where, just like in the multi-split-setup in QM, all *field configurations* are realized at the *same time* and interfere destructively and constructively.

We generalize from QM to QFT by

$$q(t) \rightarrow \phi(x) \quad (221)$$

$$p(t) \rightarrow \Pi(x) \quad (222)$$

$$j(t) \rightarrow j(x). \quad (223)$$

Herein, the arguments x of the fields and source are understood as spacetime arguments, i.e., ϕ etc. depend on spatial coordinates and time.

The main results from QM generalize and we obtain a generating functional

$$Z[j] = \int \mathcal{D}\Pi(x) \mathcal{D}\phi(x) e^{i \int d^4x (\Pi(x) \dot{\phi}(x) - (1-i\varepsilon)H + j(x)\phi(x))}. \quad (224)$$

Because H is quadratic in $\Pi(x)$,

$$H = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi), \quad (225)$$

we can perform the Gaussian functional integral over $\Pi(x)$ and obtain

$$Z[j] = \int \mathcal{D}\phi(x) e^{i \int d^4x (\mathcal{L}(\phi) + j(x)\phi(x))} = \int \mathcal{D}\phi e^{iS + \int j\phi}. \quad (226)$$

The generating functional generates *correlation functions*, which are expectation values of the field at different spacetime points,

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = i^n \frac{\delta}{\delta j(x_1)} \dots \frac{\delta}{\delta j(x_n)} Z[j] = \int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) e^{iS + \int j\phi}. \quad (227)$$

These correlation functions tell us about the expectation value of the field in the vacuum and the *correlations* between field values at different points. When we remember that in the canonical formalism, $\phi(x)$ acting on the vacuum generates a particle, we can see that the correlators give us information on the amplitude for a particle to propagate from one point to another (for $\langle \phi(x)\phi(y) \rangle$) and the amplitudes for three, four etc particles to interact. Therefore, these correlation functions will later become important, when we describe particle scattering in quantum field theory. We will come back to them later and develop a better and more detailed understanding of them.

Let us contrast classical field theory and quantum field theory: Classically, a single field configuration $\phi_{\text{class}}(x)$ is realized, for which

$$\left. \frac{\delta S}{\delta \phi} \right|_{\phi=\phi_{\text{class}}} = 0. \quad (228)$$

In QFT, all field configurations are realized at the same time. Each comes with a complex phase factor $e^{iS[\phi]}$, that results in destructive/constructive interference between field configurations. When S varies slowly, the interference is constructive, because “neighboring” configurations have nearly the same phase factor e^{iS} . Thus, we expect that a main contribution to the path integral actually comes from the classical field configurations. In contrast, field configurations far from the classical ones have a quickly varying phase factor e^{iS} , which oscillates between $+1$ and -1 quickly, and thus these cancel out or interfere destructively.

Overall, the path integral formalism provides a different, and highly useful, intuition for the physics of QFT, as well as a powerful formalism.

5 The quantum effective action $\Gamma[\Phi]$ and interacting quantum field theory

(See Gelis, Ryder, Padmanabhan ...)

In QFT, classical field configurations lose their meaning, because no *single* field configuration is ever realized (see the discussion above). However, the expectation value $\langle \phi \rangle = \int \mathcal{D}\phi \phi e^{iS[\phi]}$ has physical meaning, as do the n -point correlation functions $\langle \phi(x_1) \dots \phi(x_n) \rangle$.

Therefore, just like S produces the classical equations of motion, we would like to have a “quantum version” that gives the equations of motion for the expectation value $\langle \phi \rangle$.

We will call this (at this stage hypothetical) object $\Gamma[\Phi]$, where we have introduced the notation $\Phi = \langle \phi \rangle$, to clarify that the argument of Γ is *not* an individual classical field configuration, because Γ must actually arise from the path integral.

Let us define

$$e^{i\Gamma[\Phi] + i \int d^4x j(x)\Phi(x)} = \int \mathcal{D}\phi e^{iS[\phi] + i \int d^4x j(x)\phi(x)}, \quad (229)$$

because then $\Gamma[\Phi]$ appears analogously to $S[\phi]$ (in a complex exponential) and, if we would “switch off” quantum fluctuations, i.e., only have a contribution from the classical field configurations, then $\Gamma[\Phi] = S[\Phi]$.

In fact, this definition implies that

$$\Gamma[\Phi] = -i \ln Z[j] - \int d^4x j(x)\Phi(x), \quad (230)$$

i.e. $\Gamma[\Phi]$ is the *Legendre transform* of $\ln Z[j]$.

Mini-Exercise 12. Check that this expression for $\Gamma[\Phi]$ follows from our definition above.

Solution.

$$\begin{aligned} e^{i\Gamma[\Phi] + i \int d^4x j(x)\Phi(x)} &= \underbrace{\int \mathcal{D}\phi e^{iS[\phi] + i \int d^4x \phi(x)j(x)}}_{Z[j]} \\ \implies \ln \left(e^{i\Gamma[\Phi]} e^{i \int d^4x j(x)\Phi(x)} \right) &= \ln Z[j] \\ i\Gamma[\Phi] + i \int d^4x j(x)\Phi(x) &= \ln Z[j] \\ \implies \Gamma[\Phi] &= -i \ln Z[j] - \int d^4x j(x)\Phi(x). \end{aligned}$$

From there, we obtain

$$\frac{\delta \Gamma[\Phi]}{\delta \Phi} = -j(x). \quad (231)$$

This is the *quantum analogue* of the classical equations of motion. It gives us the equations of motion for the expectation value Φ .

Effective action as Legendre transform:

Let us be a bit more precise about the definition of $\Gamma[\Phi]$ as the Legendre transform of $Z[j]$. First, we review what a Legendre transform of a function $f(x)$ is. Let the derivative of $f(x)$ be p , i.e.,

$$\frac{df}{dx} = p. \quad (232)$$

In our case, this would be a functional version of such an equation, namely $\frac{\delta}{\delta j(x)} Z[j] = \Phi$. We are now interested in obtaining a quantity that takes p as its argument, but contains the information on the function $f(x)$, i.e., we are looking for some $g(p)$ that arises from $f(x)$ and is in a precise correspondence to $f(x)$. We can derive from Eq. (232) that

$$d(xp - f(x)) = x dp + p dx - df = x dp. \quad (233)$$

Thus, by defining

$$g(p) = \sup_x (xp - f(x)), \quad (234)$$

we have that

$$\frac{dg}{dp} = x. \quad (235)$$

The supremum means that the right-hand side is evaluated at the value of x , where it takes its maximal value. This is how $g(p)$ depends only on p , and not also on x , as it would be, if we would leave out the supremum.

In our case, $g(p)$ is $\Gamma[\Phi]$, and we note that Eq. (230) should properly read

$$\Gamma[\Phi] = \sup_j \left(-i \ln Z[j] - \int d^4x j(x) \Phi(x) \right). \quad (236)$$

Let us finish this discussion with an example of the Legendre transform of a function. We consider $f(x) = x^2$. Then $px - f(x)$ is maximized at $x = p/2$, and thus $g(p) = (x \cdot p - f(x)) \Big|_{x=p/2} = p^2/2 - p^2/4 = p^2/4$.

Comment: There's lots more to say about Γ and we will come back to it in QFT II to develop our formal and physical understanding of it further.

5.1 Effective potential in scalar $\lambda\phi^4$ theory

(See Gelis, Ryder, Peskin/Schröder, Padmanabhan)

We are now ready to take a first look at an interacting theory. We will learn about the effects of quantum fluctuations and will also encounter UV divergences again that we have to deal with.

We consider the simplest interacting theory, namely $\lambda\phi^4$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 - \frac{\lambda}{12} \phi^4. \quad (237)$$

What is the physical meaning of the $\phi(x)^4$ term? Remember that $\phi(x)$, understood as an operator, when acting on the vacuum, produces a one-particle state with the particle localized at x . Thus, if we act with $\phi(x)$ four times, we describe four particles, all localized at x . This is clearly what we need in order to describe *interacting* particles, because, to describe interactions, we must have several particles, not just a single one.

Note also that the interaction is *local*, i.e., particles interact at a single point in spacetime. (An example for a nonlocal interaction term would be $\int d^4x \int d^4y \phi(x)^2 \phi(y)^2$. There is no QFT with such interactions, because causality would be impossible to maintain in such a theory.) Local interactions are what we observe in nature (e.g., at particle colliders, interactions happen when particles meet at a point, they do not happen over a distance) and thus we use local interaction terms in Lagrangians.

We exclude the ϕ^3 -term by a $\phi \rightarrow -\phi$ symmetry ($\mathbb{Z}_2$ symmetry) and we neglect other interactions (ϕ^6 , $\partial_\mu \phi \partial^\mu \phi \phi^2$ etc.) for now. ϕ^4 is an interaction term, because it yields a non-linear term in the equations of motion, i.e., instead of a freely propagating wave-like solution, we have *self-interactions*. In terms of the corresponding particles, we can have *scattering* of the particles off each other.

The path integral for an interacting theory is complicated. It is no longer, as it would be for the free theory, a Gaussian integral that we can do exactly. Therefore, we will consider an expansion for it, in which the leading non-trivial correction is also obtained from a term quadratic in the fields.

We expand around $\phi_{\text{cl}} = \langle \phi \rangle$. Because we expect that the ground state of the theory respects translation invariance, $\phi_{\text{cl}} = \text{const.}$ This constant may be zero, but, and this will be the more interesting case, it may also be non-zero. In this case, the $\mathbb{Z}_2$ -symmetry is broken *spontaneously*, i.e., there is a symmetry of the Lagrangian that the ground state breaks. This also happens in the Higgs sector of the Standard Model and is the mechanism through which the elementary particles in the SM acquire their mass⁹.

In $Z[j]$, we can perform a *shift* of the integration variable,

$$Z[j] = \int \mathcal{D}\phi e^{i(S[\phi] + \int j\phi)} \underset{\phi = \phi_{\text{cl}} + \varphi}{=} \int \mathcal{D}\varphi e^{iS[\phi_{\text{cl}} + \varphi] + i \int j(\phi_{\text{cl}} + \varphi)}. \quad (238)$$

Now, we can expand S in φ . Intuitively, we can imagine that we are aiming at calculating the impact of small quantum fluctuations around ϕ_{cl} , i.e., we account for the effect of field configurations close to ϕ_{cl} .

We set $j = \frac{\delta S}{\delta \phi} \Big|_{\phi = \phi_{\text{cl}}}$, so that the source only sources the classical field. Then,

$$\begin{aligned} & S[\phi_{\text{cl}} + \varphi] + \int d^4x j(\phi_{\text{cl}} + \varphi) \\ &= S[\phi_{\text{cl}}] + \int d^4y \frac{\delta S}{\delta \phi(y)} \Big|_{\phi = \phi_{\text{cl}}} \varphi(y) + \frac{1}{2} \int d^4y d^4z \varphi(y) \frac{\delta^2 S[\phi]}{\delta \phi(y) \delta \phi(z)} \Big|_{\phi = \phi_{\text{cl}}} \varphi(z) \\ & \quad + \dots + \int d^4x j(\phi_{\text{cl}} + \varphi). \end{aligned} \quad (240)$$

Because of $j = \frac{\delta S}{\delta \phi} \Big|_{\phi = \phi_{\text{cl}}}$, the term linear in φ and the source term with φ cancel. We evaluate the $\delta^2 S$ term as a mini exercise.

Mini-Exercise 13. Evaluate

$$\frac{1}{2} \int d^4y d^4z \varphi(y) \frac{\delta^2 S[\phi]}{\delta \phi(y) \delta \phi(z)} \Big|_{\phi = \phi_{\text{cl}}} \varphi(z). \quad (241)$$

⁹The QCD bound states, such as the photon and neutron, acquire most of their mass through another spontaneous symmetry breaking due to the strong-coupling regime that QCD enters at low energies.

Solution. We obtain ($\square = \partial_\mu \partial^\mu$)

$$\begin{aligned}
& \frac{\delta}{\delta\phi(y)} \frac{\delta}{\delta\phi(z)} \int d^4x \left(-\frac{1}{2}\phi(x)\square\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{12}\phi^4(x) \right) \Big|_{\phi=\phi_{\text{cl}}} \\
&= \int d^4x \left(-\frac{1}{2}\delta^4(x-y)\square\delta^4(x-z) - \frac{1}{2}\delta^4(x-z)\square\delta^4(x-y) \right. \\
&\quad \left. - m^2\delta^4(x-y)\delta^4(x-z) - \lambda\phi^2(x)\delta^4(x-y)\delta^4(x-z) \right) \Big|_{\phi=\phi_{\text{cl}}} \\
&= -\square\delta^4(y-z) - m^2\delta^4(y-z) - \lambda\phi_{\text{cl}}^2\delta^4(y-z).
\end{aligned}$$

We can write this as $-(\square + V''(\phi_{\text{cl}}))\delta^4(y-z)$, where $V'' = \frac{d^2V}{d\phi^2}$. Thus,

$$Z[j] = \int \mathcal{D}\varphi e^{iS[\phi_{\text{cl}}] - \frac{1}{2} \int d^4y d^4z \varphi(y) [(\square + V''(\phi_{\text{cl}}))\delta^4(y-z)] \varphi(z) + i \int d^4x j\phi_{\text{cl}}} \quad (242)$$

$$\begin{aligned}
&= \underbrace{e^{iS[\phi_{\text{cl}}] + i \int d^4x j\phi_{\text{cl}}}}_{\substack{\text{independent of } \varphi, \\ \text{can therefore be pulled out} \\ \text{of the path integral.}}} \int \mathcal{D}\varphi e^{-\frac{1}{2} \int d^4y \varphi(y) (\square + V''(\phi_{\text{cl}})) \varphi(y)}. \quad (243)
\end{aligned}$$

Now we need that (for a proof, see the exercises)

$$\int \mathcal{D}\varphi e^{-\frac{1}{2} \int d^4x \varphi(x) A \varphi(x)} = (\det A)^{-\frac{1}{2}}, \quad (244)$$

$$\det A = e^{\text{tr} \ln A}, \quad (245)$$

$$(\det A)^{-\frac{1}{2}} = e^{-\frac{1}{2} \text{tr} \ln A}. \quad (246)$$

This helps us, because this expression also continues to hold if the "matrix" A is infinitely large, i.e., we use that

$$\int d^4x \int d^4y \varphi(x) (\square + V'') \delta^4(x-y) \varphi(y) = \int d^4x \varphi(x) A(x,y) \varphi(y) \leftarrow \sum_i \sum_j \varphi(x_i) A_{ij} \varphi(x_j). \quad (247)$$

Thus,

$$Z[j] = e^{iS[\phi_{\text{cl}}] + i \int j\phi_{\text{cl}}} e^{-\frac{1}{2} \text{tr} \ln (-i(\square + V''(\phi_{\text{cl}})))} + \dots \quad (248)$$

and finally, we obtain the expression for the *one-loop effective action* for a constant ϕ_{cl} :

$$\Gamma[\phi_{\text{cl}}] = S[\phi_{\text{cl}}] + \int j\phi_{\text{cl}} + \frac{i}{2} \text{tr} \ln (\square + V''(\phi_{\text{cl}})), \quad (249)$$

where we first calculate $-i \ln Z[j]$ to obtain Γ , and where we can drop the "i" inside the $\ln$, because it only produces a constant addition to Γ which is field-independent and therefore irrelevant for the physics.

This expression makes sense in that $\Gamma[\phi_{\text{cl}}] = S[\phi_{\text{cl}}]$ to leading order. If we reinstate $\hbar$, then the $\text{tr} \ln$ -term is $\sim \hbar$, so it encodes the leading-order quantum correction. This term is called the 1-loop term, because the tr implies an integration over momenta (the eigenvalues of $\square$) of quantum fluctuations. When we introduce the expansion in terms of Feynman diagrams later, we will see that we will denote such an integration by a closed loop.

In the case of $\phi_{\text{cl}} = \text{const}$ that we consider here, $\Gamma[\phi_{\text{cl}}] = \int d^4x V_{\text{eff}}[\phi_{\text{cl}}]$, because the kinetic term vanishes for this case. V_{eff} is called the *effective potential*. Our remaining task is to evaluate

$$\frac{i}{2} \text{tr} \ln (\square + V''(\phi_{\text{cl}})) = \frac{i}{2} \int d^4x \int \frac{d^4p}{(2\pi)^4} \ln \left(-p^2 + \underbrace{m^2 + \lambda\phi_{\text{cl}}^2}_{\text{constants}} \right). \quad (250)$$

$\int d^4x$ just becomes a factor of spacetime volume that also appears before V_{eff} . These drop out of V_{eff} . To evaluate the integral, we do several things:

- we note that the integral is divergent at large p^2 . This is again a consequence of us having simply assumed that our QFT is valid up to arbitrarily small distances (i.e., arbitrarily high momenta).
- we point out that m^2 , which we have been calling “mass”, does not actually correspond to a measurable quantity. Rather, $\left. \frac{\partial^2}{\partial \phi_{\text{cl}}^2} V \right|_{\phi_{\text{cl}}=0} = m_{\text{Phys}}^2$ is the actual mass that is associated to the field. If we could turn off $\hbar$, then m_{Phys}^2 would correspond to the mass, but in nature, we *cannot* turn off $\hbar$ (or, more physically, we cannot turn off quantum fluctuations).

These two observations combined lead us to the following ideas:

- 1) The momentum integral should be regularized. The most intuitive way is through a cutoff Λ in the momentum integral, although there are other ways of regularizing, e.g., dimensional regularization, that we encounter later.
- 2) The cutoff (or more generally the regularization) just parametrizes our ignorance of the correct physics at high momenta (small distances).

The *physical results* of measurements (e.g., of the mass of a particle or the strength with which particles interacting) do not care about whether or not we do or do not understand the UV physics. Therefore,

$$\left. \frac{\partial^2}{\partial \phi_{\text{cl}}^2} V_{\text{eff}}(\phi_{\text{cl}}) \right|_{\phi_{\text{cl}}=0} = m_{\text{Phys}}^2 \quad (\star 1)$$

$$\left. \frac{\partial^4}{\partial \phi_{\text{cl}}^4} V_{\text{eff}}(\phi_{\text{cl}}) \right|_{\phi_{\text{cl}}=0} = 2\lambda_{\text{Phys}} \quad (\star 2)$$

must not depend on the cutoff Λ (or whatever other parameter determines our regularization). To achieve $(\star 1)$ and $(\star 2)$ is independent from Λ , we are led to the conclusion that m^2 and λ must be functions of Λ . We are free to make them that, because they are not physical parameters that we could measure.

Therefore, we will implement the procedure of regularization and renormalization. Note that “renormalization” is a misnomer, because it suggests that we have already normalized the theory and now we need to “re”-normalize all parameters by huge (potentially even infinite) shifts. This is a confusing view of the actual procedure, where we normalize the measurable parameters once.

In practise, we now need to regularize. There are different methods of regularization, e.g., zeta-function regularization (which is probably the most abstract of the regularization methods typically used), dimensional regularization (which we will encounter and use later in the course, and which is

a very common method in high-energy physics) and cutoff-regularization. We already encountered cutoff-regularization when we considered the Casimir effect, and we will use it again here. Arguably, in cutoff-regularization, it is easiest to understand what is going on.

To perform the regularization, we perform a *Wick-rotation* to Euclidean signature, i.e., we send

$$t \rightarrow i\tau, \quad (251)$$

where τ is Euclidean "time", i.e., the "time" coordinate in a space with a metric with all negative signs. Then, $p^2 \rightarrow -p_E^2$, and

$$\int \frac{d^4 p}{(2\pi)^4} \ln(-p^2 + m^2 + \lambda\phi_{\text{cl}}^2) \rightarrow \int \frac{d^4 p_E}{(2\pi)^4} \ln(p_E^2 + m^2 + \lambda\phi_{\text{cl}}^2). \quad (252)$$

The reason for doing a Wick-rotation is that we can now successfully introduce a cutoff, i.e., limit $p_E^2 < \Lambda^2$. In Minkowski signature, $p^2 < \Lambda^2$ does not effectively work as a cutoff, because $p^2 = p_0^2 - \vec{p}^2$, and thus $p^2 < \Lambda^2$ still allows arbitrarily high energies p_0^2 , as long as they come with an arbitrarily high spatial momentum $\vec{p}_i^2$.

Thus,

$$\begin{aligned} \int \frac{d^4 p_E}{(2\pi)^4} \ln(p_E^2 + m^2 + \lambda\phi_{\text{cl}}^2) &= \int_0^\infty \frac{dp_E^2}{32\pi^2} p_E^2 \ln(p_E^2 + m^2 + \lambda\phi_{\text{cl}}^2) \\ &\rightarrow \int_0^{\Lambda^2} \frac{dp_E^2}{32\pi^2} p_E^2 \ln(p_E^2 + m^2 + \lambda\phi_{\text{cl}}^2) \\ &\rightarrow \int_0^{\Lambda^2} \frac{dp_E^2}{32\pi^2} p_E^2 \ln\left(\frac{p_E^2 + m^2 + \lambda\phi_{\text{cl}}^2}{\Lambda^2}\right). \end{aligned} \quad (253)$$

In the first step, we have used that the integral is rotationally symmetric in momentum space. In the second step, we have introduced a regularization, that we ultimately would like to remove again, i.e., we aim at sending $\Lambda^2 \rightarrow \infty$ to recover the original integral. In the meantime, we are keeping in mind that the high-momentum part of the integral comes from a regime where we cannot be sure whether our theory actually describes nature – we have simply extrapolated our theory to arbitrarily high momenta (arbitrarily small distances). Thus, Λ^2 can be given a physical interpretation in the sense that it cuts off the part of the momentum integral, where some "new physics" may be, that our theory does not account for and therefore does not adequately describe this regime.

In the very last step, we have noted that the argument of the $\ln$ is dimensionful (which it should not be), and have therefore divided by Λ^2 . This we have achieved by subtracting $\int_0^{\Lambda^2} \frac{dp_E^2}{32\pi^2} p_E^2 \ln(\Lambda^2)$ from the previous expression. We are free to do this, because there is no dependence on the *field* in this. We are just subtracting a field-independent constant, which does not contribute to the equations of motion (and just shifts the value of the ground-state energy, but has no effect on measurements, because we can only measure differences in energies).

Now we have an expression that we can evaluate and interpret. We will learn several important consequences of quantum fluctuations, that are not specific to this particular theory, but rather generic features of many QFTs.

We obtain

$$\int_0^{\Lambda^2} \frac{dp_E^2}{32\pi^2} p_E^2 \ln \left(\frac{p_E^2 + m^2 + \lambda\phi_{\text{cl}}^2}{\Lambda^2} \right) = \frac{1}{4 \cdot 32\pi^2} \left[\Lambda^2 (2m^2 - \Lambda^2 + 2\lambda\phi_{\text{cl}}^2) + 2\Lambda^4 \ln \left(\frac{m^2 + \Lambda^2 + \lambda\phi_{\text{cl}}^2}{\Lambda^2} \right) - 2(m^2 + \lambda\phi_{\text{cl}}^2)^2 \left(\ln \left(\frac{m^2 + \lambda\phi_{\text{cl}}^2 + \Lambda^2}{\Lambda^2} \right) - \ln \left(\frac{m^2 + \lambda\phi_{\text{cl}}^2}{\Lambda^2} \right) \right) \right]. \quad (254)$$

From there, by collecting all prefactors, and adding the classical contribution, we obtain the effective potential

$$V_{\text{eff}}(\phi_{\text{cl}}) = \frac{1}{4 \cdot 32\pi^2} \left[\Lambda^2 (2m^2 - \Lambda^2 + 2\lambda\phi_{\text{cl}}^2) + 2\Lambda^4 \ln \left(\frac{m^2 + \Lambda^2 + \lambda\phi_{\text{cl}}^2}{\Lambda^2} \right) - 2(m^2 + \lambda\phi_{\text{cl}}^2)^2 \left(\ln \left(\frac{m^2 + \lambda\phi_{\text{cl}}^2 + \Lambda^2}{\Lambda^2} \right) - \ln \left(\frac{m^2 + \lambda\phi_{\text{cl}}^2}{\Lambda^2} \right) \right) \right] + \frac{m^2}{2} \phi_{\text{cl}} + \frac{\lambda}{12} \phi_{\text{cl}}^4. \quad (255)$$

This is a complicated and somewhat lengthy expression, so there is lots to unpack. In fact, there are also several important physical aspects to learn from this expression that we will go through in detail below:

1. We see that the effective potential contains terms that have the same field dependence as the classical terms, but come with prefactors that will diverge if we take $\Lambda^2 \rightarrow \infty$. One example is the first term in the first line of Eq. (255), which is $2\Lambda^2 \lambda \phi_{\text{cl}}^2$. These are the terms that we will have to deal with through renormalization and that have caused some confusion about QFT, in particular in the earlier years of the development of QFT.
2. We also see that, upon expanding the ln-terms, we will obtain higher powers of ϕ_{cl} , e.g., ϕ_{cl}^6 or ϕ_{cl}^8 . This is another *generic feature of QFTs: quantum fluctuations generate new interactions*. This result is not specific to ϕ^4 theory, but appears in virtually all other QFTs. A well-known, and physically really interesting, example, is Quantum Electrodynamics: one of the phenomenologically crucial features of classical electrodynamics is that electromagnetic waves do not interact with each other, i.e., the equations of motion are linear, and accordingly photons do not interact. If this was different, and the equations would have strong non-linearities, we would not be able to observe distant galaxies (nor would one be able to see a black-board from across a lecture room, if the non-linearity was really strong). It turns out that when we integrate quantum fluctuations of the electron, a four-photon-interaction term is generated. Therefore, the quantum version of electrodynamics is fundamentally different from the classical version – at least in principle. In practise, we are saved by the fact that the photon-photon-interaction term comes with a tiny prefactor, such that one needs extremely high electromagnetic fields to observe the non-linearities. High-intensity lasers are constructed with the goal (among other goals, of course), to observe these terms for the first time.

The corresponding action, encoding these terms, can be obtained from Quantum Electrodynamics along exactly the same line as our calculation for the scalar field and is called the *Euler-Heisenberg* effective action.

3. It is not obvious from the above expression, but, after we have taken care of the divergences, the resulting potential is generically no longer minimized at the point $\phi_{\text{cl}} = 0$. Instead, a

non-zero expectation value of ϕ permeates the vacuum. This is called *spontaneous symmetry breaking* (i.e., the ground state does not realize a symmetry of the action; in this case, the $\mathbb{Z}_2$ symmetry of the action, under which $\phi \rightarrow -\phi$, is not realized by the ground-state, because the only field configuration that realizes that symmetry is $\phi = 0$, but we will have that $\phi_{\text{cl}} \neq 0$ at the minimum of the effective potential.

Spontaneous symmetry breaking is a crucial ingredient of the Standard Model of particle physics, where the Higgs sector has a spontaneously broken symmetry. The resulting non-zero vacuum expectation value of the Higgs field that permeates the vacuum results in mass-generation for the fermions as well as some of the gauge bosons of the Standard Model.

Here, we will see that quantum fluctuations can have the effect to spontaneously break a symmetry that is realized classically.

To see all these results, we will now analyze the ϕ_{cl}^2 , ϕ_{cl}^4 and ϕ_{cl}^6 terms separately and then turn to the analysis of the full expression for the finite part of the effective potential.

We first note that any terms that are of zeroth order in the field, whether they are divergent or not, can simply be ignored. They contribute to a (possible infinite) shift of what we declare to be "zero energy", but (under the assumption of gravity not existing, that we operate under in this course), this shift can be ignored, because we can only measure energy differences.

To isolate individual powers of ϕ_{cl} , we expand the logarithmic terms as follows, which we can do, because we assume that $\Lambda^2 \gg m^2, \phi_{\text{cl}}^2$.

$$\ln \left(\frac{\Lambda^2 + m^2 + \lambda \phi_{\text{cl}}^2}{\Lambda^2} \right) = \ln \left(1 + \frac{m^2 + \lambda \phi_{\text{cl}}^2}{\Lambda^2} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{m^2 + \lambda \phi_{\text{cl}}^2}{\Lambda^2} \right)^n, \quad (256)$$

as well as

$$\ln \left(\frac{m^2 + \lambda \phi_{\text{cl}}^2}{\Lambda^2} \right) = \ln \left(\frac{m^2}{\Lambda^2} \right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\lambda \phi_{\text{cl}}^2}{m^2} \right)^n. \quad (257)$$

Thus we obtain

$$\begin{aligned} V_{\text{eff}} \Big|_{\phi_{\text{cl}}^2} &= -\frac{1}{8 \cdot 32\pi^2} \left(\Lambda^2 2\lambda + 2\Lambda^4 \frac{\lambda}{\Lambda^2} - 2 \cdot 2m^2 \lambda \frac{m^2}{\Lambda^2} - 2m^4 \frac{\lambda}{\Lambda^2} - 2 \cdot 2m^2 \lambda \ln \frac{m^2}{\Lambda^2} - 2m^4 \frac{\lambda}{m^2} \right) \\ &\quad + \frac{m^2}{2}. \end{aligned} \quad (258)$$

The terms in the first line originate from quantum fluctuations, the term in the second line is the classical contribution. We observe that *all terms in the first line are $\sim \lambda$* , i.e., interactions are necessary in order for quantum fluctuations to have an effect on the potential. Heuristically, we can imagine quantum fluctuations as virtual particles. A cloud of "virtual" particles is always there, even in the ground state, and it interacts with any real particles, and can affect their properties (e.g., their mass). However, to have such an effect, virtual and real particles need to interact with each other, and thus, there must be an interaction term in the theory for the effect to be present. We see that there are some contributions to the ϕ_{cl}^2 -term which are finite, even in the limit $\Lambda^2 \rightarrow \infty$, so the effect of quantum fluctuations that changes the potential, is there *irrespective of whether or not UV divergences are present*.

However, there are also contributions that would diverge in the limit $\Lambda^2 \rightarrow \infty$. To deal with them, we remember that *we cannot measure the classical term and the effect of quantum fluctuations separately, we can only measure their combination*. Thus, m^2 is so far an unspecified parameter.

If we thus require that

$$V_{\text{eff}} \Big|_{\phi_{\text{cl}}^2} = \frac{m_{\text{phys}}^2}{2} \phi_{\text{cl}}^2, \quad (259)$$

with a finite m_{phys}^2 (that corresponds to the value an experiment would measure), we see that we have to make the parameter m^2 dependent on Λ in such a way that their combination gives m_{phys}^2 . This, in essence, is the idea of *renormalization*, that we develop in more detail later in the course and in QFT II.

Next, we turn to the ϕ_{cl}^4 term. To isolate it from the $\ln$ -terms, it is useful to use of a slightly different expansion, namely

$$\ln \left(\frac{m^2 + \Lambda^2 + \lambda \phi_{\text{cl}}^2}{\Lambda^2} \right) = \ln \left(1 + \frac{m^2}{\Lambda^2} \right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\lambda \phi_{\text{cl}}^2}{m^2 + \Lambda^2} \right)^n. \quad (260)$$

Using this, we obtain

$$\begin{aligned} V_{\text{eff}} \Big|_{\phi_{\text{cl}}^4} = & -\frac{1}{8 \cdot 32\pi^2} \left[2\Lambda^4 \frac{\lambda^2}{m^2 + \Lambda^2} - 2\lambda^2 \left(\ln \left(1 + \frac{m^2}{\Lambda^2} \right) - \ln \frac{m^2}{\Lambda^2} \right) \right. \\ & \left. + 2m^2 \cdot 2\lambda \left(\frac{\lambda}{m^2 + \Lambda^2} - \frac{\lambda}{m^2} \right) - 2m^4 \left(-\frac{\lambda^2}{2(m^2 + \Lambda^2)^2} + \frac{\lambda^2}{2m^4} \right) \right] + \frac{\lambda}{12}. \end{aligned} \quad (261)$$

We again see that the terms from quantum fluctuations are proportional to the interaction, in this case, they are $\sim \lambda^2$. There are terms that remain finite in the limit $\Lambda^2 \rightarrow \infty$, so we see that quantum fluctuations also change the quartic term (i.e., the *strength* of the interaction), in the effective potential.

Then, there are again divergent terms. We absorb them by defining the physical interaction strength

$$V_{\text{eff}} \Big|_{\phi_{\text{cl}}^4} = \frac{\lambda_{\text{phys}}}{12}. \quad (262)$$

This is possible, because we can adjust λ to depend on Λ in such a way that the divergences in λ_{phys} cancel.

In general, we will call theories in which we can absorb divergences within the existing mass/coupling parameters *renormalizable*. This means that in such theories, the unknown physics at high momenta (beyond the cutoff), affects these couplings (in our case, mass and quartic coupling). Therefore, we cannot know the physical values m_{phys}^2 and λ_{phys} , because the UV physics that they depend on is not known. They parameterize our ignorance about the UV properties of the theory.

However, the rest of the coupling parameters (in our case, this will be λ_6 of ϕ^6 etc, as we will see below), are *calculable* and *independent of the UV physics*.

Renormalizability thus means that we have a theory with finitely many free parameters – a *predictive* theory.

Usually, in QFT courses, renormalization is discussed later, and we will also come back to discuss it in more detail. However, taking a look at it early on is useful not just to understand what the effect of quantum fluctuations – the existence of which is the key difference between a classical field theory and a quantum field theory – actually is, but also in order to check whether they result in a theory that is not predictive at the quantum level, because it has a finite number of free parameters.¹⁰

¹⁰There is much more to say and understand here, part of which we will cover in QFT II. It has to do with

Now let us consider the ϕ_{cl}^6 terms. If these are also divergent, then we are in trouble, because we have used up the free parameters of our classical action, m^2 and λ , to absorb the divergences.

We obtain

$$V_{\text{eff}}\big|_{\phi_{\text{cl}}^6} = \frac{-1}{8 \cdot 32\pi^2} \left[2\Lambda^4 \frac{\lambda^3}{3(m^2 + \Lambda^2)^3} - 2m^4 \left(\frac{\lambda^3}{3(m^2 + \Lambda^2)^3} - \frac{\lambda^3}{3(m^2)^3} \right) \right. \\ \left. - 2m^2 \cdot 2\lambda \left(-\frac{\lambda^2}{2(m^2 + \Lambda^2)^2} + \frac{\lambda^2}{2(m^2)^2} \right) - 2\lambda^2 \left(\frac{\lambda}{m^2 + \Lambda^2} - \frac{\lambda}{m^2} \right) \right] + 0. \quad (263)$$

The last term is the classical contribution, which is zero. The quantum contribution is non-zero, i.e., quantum fluctuations generate new interactions.

We observe that the limit $\Lambda^2 \rightarrow \infty$ is *finite*; there are no divergences at this order in the field. We obtain

$$V_{\text{eff}}\big|_{\phi_{\text{cl}}^6} \xrightarrow{\Lambda^2 \rightarrow \infty} -\frac{1}{12 \cdot 32\pi^2} \frac{\lambda^3}{m^2}. \quad (264)$$

Therefore, quantum fluctuations generate a *calculable* ϕ^6 interaction (and similarly, higher-order interactions). These are independent of the presence of divergences in the quadratic and the quartic term.

This result, that quantum fluctuations generate new interactions, with finite, calculable coefficients, is general and is not tied to whether or not there are divergences in the theory.

Finally, let us isolate the finite piece in the effective potential without Taylor-expanding in the field. We assume that we have previously taken care of the divergences and they result in a quadratic and quartic term with undetermined coefficients, i.e.,

$$V_{\text{eff}} = \frac{m_{\text{phys}}^2}{2} \phi_{\text{cl}}^2 + \frac{\lambda_{\text{phys}}}{12} \phi_{\text{cl}}^4 + V_{\text{eff}}\big|_{\text{finite}}. \quad (265)$$

To isolate the finite terms, we have to rewrite the $\ln$ -terms, because they contribute to both the divergences (that we absorb in m^2 and λ) and the finite terms. We write

$$\begin{aligned} \ln \left(\frac{m^2 + \lambda \phi_{\text{cl}}^2 + \Lambda^2}{\Lambda^2} \right) - \ln \left(\frac{m^2 + \lambda \phi_{\text{cl}}^2}{\Lambda^2} \right) &= \ln \left(\frac{m^2 + \lambda \phi_{\text{cl}}^2 + \Lambda^2}{m^2 + \lambda \phi_{\text{cl}}^2} \right) \\ &= \ln \left(\frac{\Lambda^2}{m^2 + \lambda \phi_{\text{cl}}^2} \right) \\ &= \ln \left(\frac{\Lambda}{\mu^2} \frac{\mu^2}{m^2 + \lambda \phi_{\text{cl}}^2} \right) \\ &= \ln \left(\frac{\Lambda^2}{\mu^2} \right) + \ln \left(\frac{m^2 + \lambda \phi_{\text{cl}}^2}{\mu^2} \right). \end{aligned} \quad (266)$$

In the final line, the first term contributes to the divergent terms and the second one to the finite term. To do this separation, we had to introduce an arbitrary scale μ^2 . This is called the Renormalization scale. We will come back to it in more detail later in the course. For now we

understanding non-renormalizable theories as effective field theories, and with understanding that renormalizable theories do not automatically make sense at all scales. Keywords that we will cover later (in QFT II), are asymptotic safety, asymptotic freedom, effective field theories, the Renormalization Group and Landau poles. This will be relevant for the quantum physics of QCD, QED and the Standard Model as a whole, as well as for numerous condensed-matter models. It will even touch on ideas for a quantum theory of gravity.

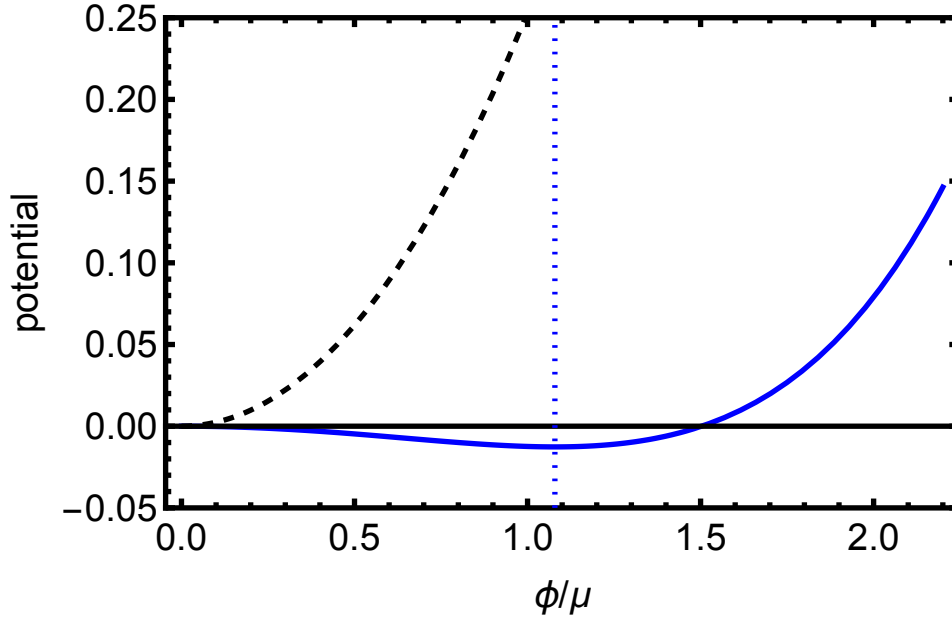


Figure 7: We show the finite part of the effective potential (for $\lambda = 0.1$ and $m^2/\mu^2 = 0.7^2$), in blue. The blue dotted vertical line indicates where the minimum of the potential lies. The black dashed line is the classical potential with the same values of mass and coupling.

can just think of it as a scale that we need to get the units right in our expressions and have dimensionless arguments of logarithms.

Thus we have

$$V_{\text{eff}}|_{\text{finite}} = \frac{1}{4 \cdot 32\pi^2} (m^2 + \lambda\phi_{\text{cl}}^2)^2 \ln \left(\frac{m^2 + \lambda\phi_{\text{cl}}^2}{\mu^2} \right). \quad (267)$$

For small enough ϕ_{cl}^2 , the $\ln$ is negative. Therefore the potential takes the form shown in Fig. 7.

Whereas the classical theory has $\phi = 0$ as a solution, quantum fluctuations introduce a *non-zero* value of ϕ_{cl} at the minimum of the potential (for some values of parameters of the theory). In those cases, the $\mathbb{Z}_2$ -symmetry of the classical action is *broken spontaneously* in the ground state. This means that the symmetry of the action is not realized by the ground state, because $\phi_{\text{cl}} = 0$, which is the only field configuration that realizes this symmetry, is not a minimum.

Instead, we can evaluate where the two possible minima lie. We take the derivative of $V_{\text{eff}}|_{\text{finite}}$ with respect to ϕ_{cl} and set it to zero. This equation determines extrema. $\phi_{\text{cl}} = 0$ is a solution, but does *not* correspond to a minimum for all values of parameters. Instead, the remaining two solutions are

$$\phi_{\text{cl}} = \pm \sqrt{\frac{\mu^2 e^{-\mu^2/2} - m^2}{\lambda}} \neq 0. \quad (268)$$

The choice between the two minima that the vacuum has to "spontaneously make", is the act of spontaneous symmetry breaking. It leads to a non-zero expectation value of the field that permeates the vacuum.