(Dod) 11.2 Signal Covariance Matrix

Now, we would like to relate theoretical predictions for the CMB and the matter distribution to the signal covariance matrix C_S.
Our predictions for the CMB are encoded in terms of the multipole coefficients C_ℓ.
As far as the matter distribution is concerned, most of the information used today is described by the power spectrum $P_{obs}(k)$.

(Dod) 11.2.1 CMB Window Functions

Following Dodelson, consider first the diagonal element of the covariance matrix

$$(C_S)_{ii} \equiv \langle s_i s_i^* \rangle \qquad \text{no sum?}$$

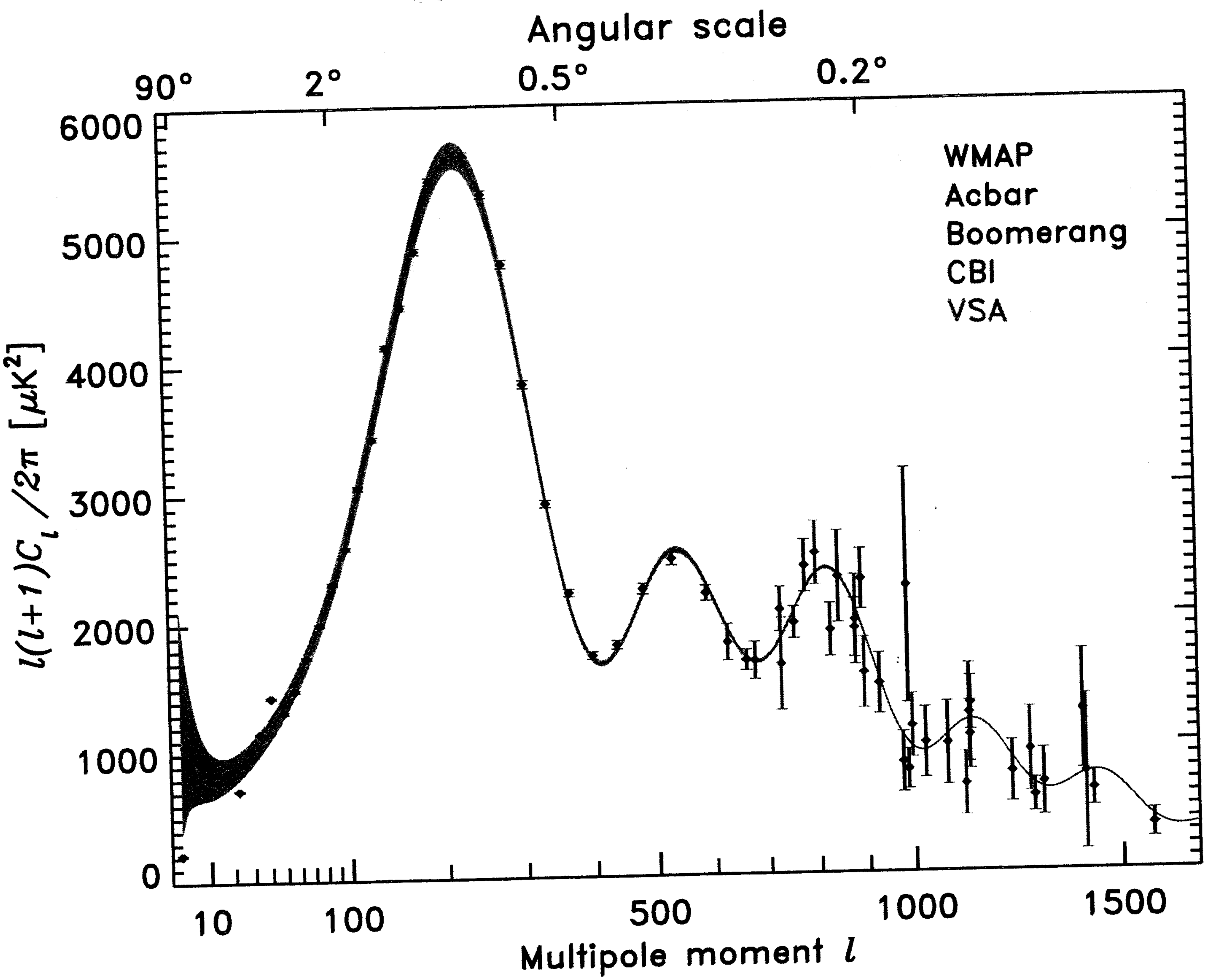
$\langle\rangle$ denotes the average over many realizations of the theoretical distribution and

$$s_i = \int d\vec{n}\; \Theta(\vec{n})\, B_i(\vec{n})$$

integrate over all directions

is the signal at pixel i.

$\Theta(\vec{n})$: underlying temperature fluctuation
Fig 11.1? $B_i(\vec{n})$: beam pattern of experiment


Angular scale
90°
2°
0.5°
0.2°
WMAP
Acbar
Boomerang
CBI
VSA
$l(l+1)C_l/2\pi$ [μK^2]
6000
5000
4000
3000
2000
1000
0
10
100
500
1000
1500
Multipole moment l

It is good practice to quote the temperature anisotropy as $\Theta(\vec{u})$ in terms of expansion coefficients $a_{\ell m}$ of spherical harmonics:

$$\Theta(\vec{u}) = T \cdot \sum_{\ell,m} Y^{\ell}_{m}(\vec{u})\, a_{\ell m}$$

where $T = 2.736$ K.

Then:

$$\frac{(C_S)_{ii}}{T^2} = \int d\vec{u}\, d\vec{u}'\, B_i(\vec{u}) B_i^*(\vec{u}') \sum_{\substack{\ell m \\ \ell' m'}} Y^{\ell}_{m}(\vec{u}) Y^{\ell'\,*}_{m'}(\vec{u}') \langle a_{\ell m} a^*_{\ell' m'} \rangle$$

statistical isotropy implies that

$$\langle a_{\ell m} \rangle = 0 ; \quad \langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$$

$$\Rightarrow \frac{(C_S)_{ii}}{T^2} = \int d\vec{u}\, d\vec{u}'\, B_i(\vec{u})\, B_i^*(\vec{u}') \quad \sum_{\ell} C_\ell \sum_{m} Y^{\ell}_{m}(\vec{u}) Y^{\ell\,*}_{m}(\vec{u}')$$

Yet (one of the many useful relations):

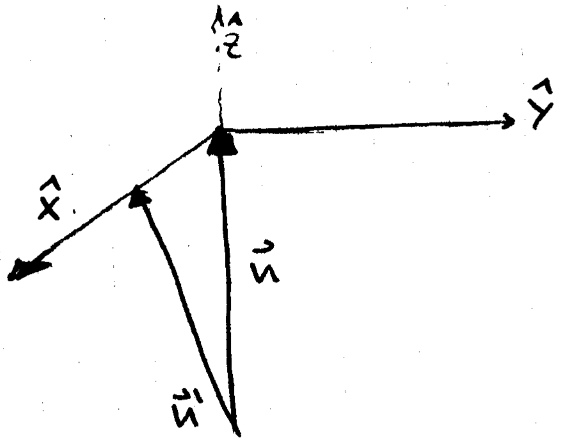
$$\sum_{m} Y^{\ell}_{m}(\vec{u}) Y^{\ell\,*}_{m}(\vec{u}') = \frac{2\ell+1}{4\pi} P_\ell(\vec{u}\cdot\vec{u}')$$

$$\frac{(C_S)_{ii}}{T^2} = \sum_{\ell} \frac{2\ell+1}{4\pi} C_\ell \quad \underbrace{\int d\vec{u}\, d\vec{u}'\, B_i(\vec{u})\, B_i^*(\vec{u}')\, P_\ell(\vec{u}\cdot\vec{u}')}$$

$$= \sum_{\ell} \frac{2\ell+1}{4\pi} C_\ell \, (W_\ell)_{ii}$$

↗ window function

As the beam pattern falls off rapidly for large separations, let us use flat sky approximation.
Consider again our unit vectors $\vec{n}$ and $\vec{n}'$.
Let's orient (for illustration) $\vec{n} \parallel \hat{z}$



$\vec{n} = (0,0,1)$; $\vec{n}' = (\varepsilon, 0, 1)$ [example]

then we can use the small angle approximation, i.e. we can define vectors $\vec{x}$ in (x,y) plane which are the transverse components of $\vec{n}$.
In our example: $\vec{x} = \vec{0}$; $\vec{x}' = (\varepsilon, 0)$
And the angle between $\vec{n}$ and $\vec{n}'$ is $\vec{x} - \vec{x}'$:

$$\vec{n} \cdot \vec{n}' = \cos(|\vec{x} - \vec{x}'|)$$

$$(W_\ell)_{ii} = \int d^2\vec{x} \int d^2\vec{x}' \, B_i(\vec{x}) B_i^*(\vec{x}') \, P_\ell(\cos[|\vec{x} - \vec{x}'|])$$

$$P_\ell(\cos[|\vec{x} - \vec{x}'|]) \to J_0(\ell |\vec{x} - \vec{x}'|) \quad ; \ell \gg 1$$
$$= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, e^{-i\ell |\vec{x} - \vec{x}'| \cos\varphi}$$

Promote ℓ to 2D vector $\vec{\ell}$ such that

$\vec{\ell}\cdot(\vec{x}-\vec{x}') = \|\ell\|\,\|\vec{x}-\vec{x}'\| \cos\varphi$ then:

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int d^2\vec{x}\, d^2x'\ B_i(\vec{x})\, B_i^*(\vec{x}')\, e^{-i\vec{\ell}(\vec{x}-\vec{x}')}$$

$$=\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int d^2\vec{x}\ B_i(\vec{x})e^{-i\vec{\ell}\vec{x}} \int d^2\vec{x}'\ B_i^*(\vec{x}')e^{i\vec{\ell}\vec{x}'}$$

So we can use Fourier transformation

$$\int d^2x\, B_i(\vec{x})e^{-i\vec{\ell}\vec{x}} \equiv \tilde{B}_i(\vec{\ell})$$

As $\vec{x}'$ part is complex conjugate of this, we get the simple result

$$(W_\ell)_{ii} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, |\tilde{B}_i(\vec{\ell})|^2$$

$\Rightarrow$ Computing Window function:

1. Calculate 2-D Fourier transform of beam pattern
2. Find angular average of square of this transform

(DoD) 11.2.2 Examples of CMB Window function

Gaussian Beam: Good approximation to many CMB experiments. Beam pattern for pixel i:

$$B_i(\vec{x}) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(\vec{x}-\vec{x}_i)^2}{2\sigma^2}\right]$$

For simplicity choose $\vec{x}_i = 0$ for window function computation:

$$\tilde{B}_i(\vec{l}) = \frac{1}{2\pi\sigma^2} \int d^2x \, e^{-i\vec{l}\vec{x}} \exp\left[-\frac{\vec{x}^2}{2\sigma^2}\right]$$

$$= e^{-l^2\sigma^2/2}$$

$\tilde{B}$ independent of direction of $\vec{l}$, so no need to average.

$$\Rightarrow (W_l)_{ii} = e^{-l^2\sigma^2}$$

Fig. 11.2 Falls off quickly at large l, i.e. ~~large~~ small distances!

$\Rightarrow$ Beam washes out structures finer than beam width!

Window functions for Galaxy Surveys

Remember that pixel i was

$$\Delta_i = \int d^3x \; \psi_i(\vec{x}) \left[\frac{n(\vec{x}) - \bar{n}(\vec{x})}{\bar{n}(\vec{x})} \right]$$

Yet the square bracket is $\delta(\vec{x})$, the fractional overdensity.

The signal covariance is then

$$(C_S)_{ij} = \langle \Delta_i \Delta_j^* \rangle \underset{\text{known}}{=} \int d^3x \, d^3x' \; \psi_i(\vec{x}) \psi_j^*(\vec{x}') \langle \delta(\vec{x}) \delta(\vec{x}') \rangle$$

and as the correlation function is defined as

$$\xi(\vec{x} - \vec{x}') \equiv \langle \delta(\vec{x}) \delta(\vec{x}') \rangle$$

we get

$$(C_S)_{ij} = \int d^3x \, d^3x' \; \psi_i(\vec{x}) \psi_j^*(\vec{x}') \; \xi(\vec{x} - \vec{x}')$$

In addition, ξ is fourier transform of $P(\vec{k})$, so

$$(C_S)_{ij} = \int d^3x \, d^3x' \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{d^3k''}{(2\pi)^3} \; \tilde{\psi}_i(\vec{k}) \tilde{\psi}_j^*(\vec{k}') \; P(\vec{k}'') \; e^{i\vec{k}\vec{x}} e^{-i\vec{k}'\vec{x}'} e^{i\vec{k}''(\vec{x} - \vec{x}')}$$

$$= \int d^3x \, d^3x' \int \frac{d^3k \, d^3k' \, d^3k''}{(2\pi)^3 (2\pi)^3 (2\pi)^3} \; \tilde{\psi}_i(\vec{k}) \tilde{\psi}_j^*(\vec{k}') \; P(\vec{k}'') \; e^{i\vec{x}(\vec{k} + \vec{k}'')} e^{-i\vec{x}'(\vec{k}' + \vec{k}'')}$$

remember that $\int d^3x \, e^{i\vec{k}\vec{x}} = (2\pi)^3 \delta(\vec{k})$

$$\Rightarrow (C_S)_{ij} = \int \frac{d^3\vec{k}}{(2\pi)^3} \; \tilde{\psi}_i(\vec{k}) \tilde{\psi}_j^*(\vec{k}) \; P(\vec{k})$$

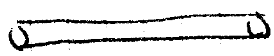
Conveniently, one defines angular part of $d^3\vec{k}$ as Window function:

$$W_{ij}(k) \equiv \int \frac{d\Omega_k}{4\pi} \tilde{\psi}_i(\vec{k}) \tilde{\psi}_j^*(\vec{k})$$

and so

$$(C_S)_{ij} = \int_0^\infty \frac{dk}{k} \left[\frac{k^3 P(k)}{2\pi^2} \right] W_{ij}(k)$$

Notice:
1. Window function is angular average over square of weighting functions like for CMB
2. Term in square brackets is $\Delta^2(k)$, the contribution to the variance per $\ln k$ interval. If this is of order unity, you have an order unity fluctuation, i.e. linear perturbation theory is surely not longer valid (just a remark)

Dodelson does now discuss two examples. A volume limited survey which observes all galaxies closer than R away from us. And a pencil beam survey which looks deep but covers only a small solid angle: 
I leave this for you to read and I only quote the results here.

For the volume limited sample, the window function is

$$W_{ii}(k) = \frac{9}{2 k k_i R^2} \int_{|k-k_i|R}^{(k+k_i)R} \frac{dy}{y} j_1^2(y)$$

Summary of (DoD) 11.2

We determined signal covariance matrix C_S for CMB and LSS. In both cases, it is a convolution of the underlying physical quantity Θ or $P(k)$ and a Window function. In principle, we would be done, because we know that for the CMB, the likelihood function is

$$\text{prob}(\vec{\Delta}\,|\,C) = \frac{(2\pi)^{-N_P/2}}{\sqrt{\det C}} \, e^{-\frac{1}{2}\vec{\Delta}^T C^{-1} \vec{\Delta}}$$

So we could scan cosmological parameter space, e.g. $(\Omega_m h^2, n_s, \Omega_b h^2, h)$ for each parameter point compute C_ℓ or $P(k)$ and from this compute C_S and $C = C_S + C_N$ and compare to experiment using prob $(\vec{\Delta}\,|\,C)$.

Yet, inverting C is costly. Can be done for COBE, impossible for WMAP in practice.

So we will now discuss methods to estimate the likelihood function.